Economic Catastrophe Bonds: Inefficient Market or Inadequate Model?

Haitao Li\textsuperscript{a} and Feng Zhao\textsuperscript{b}

ABSTRACT

In an influential paper, Coval, Jurek and Stafford (2009, CJS hereafter) argue that senior CDX tranches resemble economic catastrophe bonds—“bonds that default only under severe economic conditions.” Using a Merton structural model within a CAPM framework, CJS argue that senior CDX tranches are overpriced relative to S&P 500 index options and therefore do not sufficiently compensate investors for their inherent economic catastrophe risks. In this paper, we argue that the conclusion of CJS that senior CDX tranches are overpriced is premature. We provide compelling evidence that the CJS model is not flexible enough to capture CDX tranches prices. On the other hand, we develop a simple model that can simultaneously price CDX tranches and index options. Our results show that the CDX tranches market is actually efficient.

**JEL Classification:** C11, E43, G11

**Keywords:** CDX index and tranches, copula model, economic catastrophe risk, fat tail.

\textsuperscript{a}Stephen M. Ross School of Business, University of Michigan, Ann Arbor, MI 48109, email: htli@umich.edu.

\textsuperscript{b}School of Management, University of Texas at Dallas, Richardson, TX 75080, email: feng.zhao@utdallas.edu. We thank Ren-raw Chen, Pierre Collin-Dufresne, Robert Goldstein, and Robert Jarrow for helpful comments. We are responsible for any remaining errors.
Economic Catastrophe Bonds: Inefficient Market or Inadequate Model?

ABSTRACT

In an influential paper, Coval, Jurek and Stafford (2009, CJS hereafter) argue that senior CDX tranches resemble economic catastrophe bonds—“bonds that default only under severe economic conditions.” Using a Merton structural model within a CAPM framework, CJS argue that senior CDX tranches are overpriced relative to S&P 500 index options and therefore do not sufficiently compensate investors for their inherent economic catastrophe risks. In this paper, we argue that the conclusion of CJS that senior CDX tranches are overpriced is premature. We provide compelling evidence that the CJS model is not flexible enough to capture CDX tranches prices. On the other hand, we develop a simple model that can simultaneously price CDX tranches and index options. Our results show that the CDX tranches market is actually efficient.

JEL Classification: C11, E43, G11

Keywords: CDX index and tranches, copula model, economic catastrophe risk, fat tail.
1 Introduction

The fast growing credit derivatives markets have made it much easier to invest in the credit risk of portfolios of companies. One of the most widely traded multi-name credit derivatives is the standardized North American Investment Grade Credit Default Swap Index (CDX.NA.IG), an equally-weighted portfolio of 125 liquid credit default swap (CDS) contracts on U.S. companies with corporate debts above investment grade. Investors frequently use the CDX index to hedge a portfolio of bonds or to take positions on a basket of credit entities to speculate on changes in credit quality. CDX tranches, derivative securities with payoffs based on the losses on the underlying CDX index, are also widely used by investors who desire different risk and return properties than the CDX index.\footnote{CDX tranches are defined in terms of their loss attachment points. The 0-3, 3-7, 7-10, 10-15, and 15-30 tranches are the most popular ones in the market.} For example, even though the underlying collaterals might have low credit ratings, senior CDX tranches can have AAA rating due to the seniority of their cash flow rights.

One of the key challenges in both financial industry and academics is the pricing of CDX tranches, which depends crucially on the correlation of defaults of the underlying collaterals. The financial industry has relied heavily on the copula model of Li (2002) in setting the price of CDX tranches and other collateralized debt obligations (CDOs). The widespread adoption of the copula model is partly due to its ease of implementation. For example, in the Gaussian copula model of Li (2002), given the marginal probability of default of each individual firm, one can easily obtain the joint default probability distribution through the Gaussian copula and the prices of CDX tranches in closed-form.

The copula model, however, has several well recognized limitations. First, the parameters of the copula model do not have clear economic meanings. For example, it is rather difficult
to relate the parameters of the copula model to standard measures of credit risk, such as asset beta, market volatility, and leverage ratio. Second, the standard Gaussian copula model cannot satisfactorily capture CDX tranches prices, as evidenced by the so-called “correlation skew,” the fact that different copula correlation parameters are needed to fit different tranches. This is a clear sign of model misspecification because if the copula model is correctly specified, then the copula correlation should be the same for all CDO tranches. Finally, though various extensions to the Gaussian copula model have been proposed to capture the “correlation skew,” without a clear economic underpinning these extensions appear to be ad hoc and might overfit the data.

In an influential paper, Coval, Jurek and Stafford (2009) develop a structural model for pricing CDX tranches. They explicitly model the capital structure of a firm and assume that default happens if the firm asset value hits some exogenous default boundaries as in Merton (1974). While standard structural models do not explicitly distinguish between systematic and idiosyncratic risks, the CJS model assumes that the firm asset return follows the CAPM and that defaults of different firms are correlated due to their exposures to the same systematic factor. Though structural models have clear economic meaning, their implementations are much more complicated than that of the copula model. They tend to have a large number of parameters, require data from other markets, and lack the analytical tractability of the copula model. CJS argue that senior CDX tranches resemble economic catastrophe bonds—“bonds that default only under severe economic conditions.” CJS show that while their model can jointly price S&P 500 index options and CDX index, it cannot capture the prices of CDX tranches. They conclude that senior CDX tranches are overpriced and do not sufficiently compensate investors for the economic catastrophe risks they bear.

While CJS interpret their results as evidence of mispricing of economic catastrophe risks embedded in CDX tranches, it could also be evidence that either the CJS model is misspecified
or the markets of equity, index option, CDX index and tranches are not well integrated. In this paper, we re-examine the pricing of CDX tranches using the same data under the same modeling framework of CJS. We believe this is a very important issue in both economics and finance for several reasons.

First, the CDX index is one of the most liquid credit derivatives, even more liquid than most single-name CDS contracts. Many investors reply on CDX tranches to extract market expectations on default correlation for hedging or speculation purpose. Mispriced CDX tranches could easily lead to biased expectations and sub-optimal investment decisions. Second, as pointed by Collin-Dufresne, Goldstein, and Yang (2010, CGY hereafter), “traders in the CDX market are typically thought of as being rather sophisticated. Thus, it would be surprising to find them accepting so much risk without fair compensation.” Finally, recent papers by Longstaff and Rajan (2008), Azizpour and Giesecke (2008), and Eckner (2008) find that CDX index and tranches are consistently priced under reduced-form models, and conclude that these securities are reasonably efficiently priced. If CDX tranches are mis-priced, then it would imply some of these reduced-form models, especially models based on the top-down approach, would be invalid, because they reply on accurate CDX tranches prices to calibrate model parameters.

We first show that the structural approach has a simplified representation as the standard copula model. In particular, we show that the copula correlation measures the fraction of the total variance of firm asset return due to the systematic factor under the structural approach. We refer to this fraction as the variance ratio for the rest of the paper. Our observation greatly simplifies the implementations of structural models: If we interpret the copula correlation as the variance ratio, then the structural approach can be implemented in similar ways as the copula model. Since the copula correlation measures the variance ratio, it should depend on the same factors that drive the beta and the systematic and idiosyncratic volatilities. Moreover, since the betas of
most companies tend to increase and the relative importance of the systematic and idiosyncratic
volatilities tend to change during crisis, our approach suggests that the copula correlation should
be time varying to capture reality. Therefore, in our paper we implement structural models in the
same way as the copula model with time varying correlation. Our method shares the advantages
of both approaches and at the same time avoids their shortcomings.

We then provide a detailed analysis on the limitations of both the implementations and the
structure of the CJS model. The CJS model has several important ingredients. First, they cali-
brate the beta and volatility of the idiosyncratic factor by matching the average beta and pairwise
correlation of observed equity returns of 125 firms underlying the CDX index. Second, they as-
sume the idiosyncratic factor follows a Gaussian distribution. Third, they allow the systematic
factor to follow a skewed distribution and back it out from prices of S&P 500 index options.

We show empirically that each of the three ingredients could lead to significant pricing errors of
CDX tranches. We first show that the way CJS calibrate the beta and volatility of the idiosyncratic
factor introduces significant biases in CDX tranche prices. Following the same procedure as CJS
and using their model parameters, we obtain huge pricing errors of the CDX tranches: The
RMSE of percentage pricing errors for five year CDX tranches is 1,584.\%! On the other hand, if
we implement the CJS model as a copula model and estimate the copula correlation from prices
of CDX tranches, we obtain much smaller pricing errors of the CDX tranches: The RMSE of
percentage pricing errors is 50\%. For the purpose of pricing CDX tranches, all we need is the
variance ratio but not the individual components of the systematic and idiosyncratic volatility.
Therefore, the copula approach is more efficient than the structural approach, which has to
estimate the individual components of the total variance. Moreover, the pricing of CDX tranches
requires the variance ratio under the risk-neutral measure, whereas the estimates in the CJS paper
are under the physical measure. Finally, the CJS approach relies on the extra assumption that
the equity market is integrated with the credit market.

We then show that the Gaussian distribution of the idiosyncratic factor under the CJS model could also introduce significant biases in CDX tranches prices. CJS argue that it is reasonable to assume that the idiosyncratic factor follows a Gaussian distribution because it describes the conditional return given the systematic factor. However, we show that an idiosyncratic factor that follows fat-tailed distribution leads to significantly smaller pricing errors for CDX tranches. In our copula implementation of the CJS model with a Gaussian idiosyncratic factor, the RMSE of percentage pricing error is 50%. However, if we allow the idiosyncratic factor to follow a Student-\(t\) distribution with degree of freedom of 3, the RMSE is reduced to about 33%. This result is consistent with that of CGY (2010) who show under a structural model with stochastic volatility and jumps, it is important to introduce jumps in the dynamics of idiosyncratic factor to price CDX tranches.

Finally, we show that the functional form CJS use to back out the distribution of the systematic factor from index options is not flexible enough to capture CDX tranches prices. CJS obtain (i) the distribution of the systematic factor from index option prices and (ii) the idiosyncratic factor and debt equity ratio from individual equity prices and CDX index spreads. Then they price CDX tranches based on these calibrated parameters. In some sense, they are doing an out-of-sample analysis, using a model calibrated from index option and CDX index markets to price CDX tranches. Since their model can price index options and CDX index jointly but not CDX tranches, they argue that the CDX tranches are mis-priced. We show, however, that the CJS model is so inflexible that it cannot even capture the CDX tranches prices in sample! Specifically, we retain the functional form of the systematic factor in the CJS model, but instead of calibrating the parameters from index option prices, we obtain the parameters by directly fitting the CDX tranches prices. We also calibrate the copula correlation from CDX tranches prices and maintain
a Gaussian idiosyncratic factor. Even though the model is fitted using in sample data, the RMSE of percentage pricing errors is still as high as 45%.

Collectively, our analysis demonstrates from three different aspects the limitations of the CJS model. Therefore, the conclusion of CJS that the CDX tranches are overpriced seems to be premature. Since market efficiency tests are always a joint test of efficient market and good model, it is really difficult to draw any inference on market efficiency based on a model that cannot even capture the CDX tranches prices in sample. A more reasonable conclusion seems to be that the CJS model is probably misspecified. To determine whether CDX tranches are fairly priced, we still need to answer the question whether any model can simultaneously price index options and CDX tranches. Despite the fact that the CJS model is misspecified, if we cannot find a model that can jointly price index options and CDX tranches, then there is still the possibility that either the two markets are not well integrated or one of them is not efficient.

We develop a simple copula model that can jointly price index options and CDX tranches very well. We obtain our model by simply modifying the distributional assumptions of the systematic and idiosyncratic factors of the CJS model. CJS back out the systematic factor from index option prices based on a specific functional form and assume the idiosyncratic factor follows Gaussian distribution. In contrast, we allow both the systematic and idiosyncratic factors to follow Student-\(t\) distribution in one version of our model. It has been widely recognized that fat tails in both the systematic and idiosyncratic factors are needed to capture CDX tranches prices. The first version of the model is simply the double-\(t\) model of Hull and White (2004). Due to the volatility skew for index options, we also consider a skewed-\(t\) model, in which the systematic factor follows a skewed-\(t\) distribution of Hansen (1994) and the idiosyncratic factor follows the Student-\(t\) distribution.

We implement our model following the copula approach. Instead of separately modelling the systematic and idiosyncratic volatilities as in CJS, we focus on the variance ratio, or the copula
correlation, which is crucial for pricing CDX tranches. Following the approach of CJS, we assume that all the underlying firms of CDX index are homogeneous with the same default probability. We back out the marginal probability of default from the CDX index. As a result our model perfectly fits the CDX index by design. Taking the marginal probability of default as an input, we price the CDX tranches based on our model. Following the daily calibration approach of CJS, we choose the copula correlation and the degree of freedom of the Student-t distribution of both factors to minimize the daily percentage pricing errors of index options and CDX tranches at five year maturity, thus allowing all three important parameters to be time varying.\(^2\)

Our empirical results show that our model can capture the daily prices of the index options and CDX tranches very well. The estimated parameters show that the copula correlation and the degrees of freedom need to vary dramatically over time to capture CDX tranches prices. When we fit the double-t model to prices of CDX tranches, we obtain average RMSE of about 7.5\% for five year CDX tranches. When we fit the double-t model to prices of both CDX tranches and index options, we obtain average RMSE of about 7.7\% for CDX tranches and 7.8\% for index options. The double-t model cannot fully capture the volatility skew of index options. Finally, we fit the skewed-t model to prices of both CDX tranches and index options. We obtain average RMSE of about 7.85\% for CDX tranches and 0.96\% for index options. Hence, the skewed-t distribution is needed mainly for pricing index options.

We emphasize that the structure of our model is almost exactly the same as that of CJS. The only differences are the distributional assumptions of the systematic and idiosyncratic factors and the way we calibrate the copula correlation. Though the CJS model has huge pricing errors

\(^2\)Intuitively, time-varying systematic tail risk can change the value of a portfolio even there is no change in the loss distribution. For portfolios with zero exposures to the systematic factor, time-varying idiosyncratic tail risk can affect their values without changing the loss distribution.
for CDX tranches, our model can price both CDX tranches and index options very well. We model both factors using Student-t distribution for convenience. There are many other fat-tailed distributions we can use, which could lead to potentially even smaller pricing errors. Therefore, the fact that the simple changes we make to the CJS model can lead to such dramatic differences in pricing performance is a strong indication that the conclusion of CJS that CDX tranches are mispriced is not a robust result.

Our analysis complements the important paper of CGY (2010), who also study the relative pricing of index options and CDX tranches. They develop a structural model with stochastic volatility, stochastic dividend yield, and correlated jumps in market return, volatility and dividend yield, and show that the model can price index options and CDX tranches reasonably well. Our model differs from the CGY model in several important ways. First, since our model stays strictly within the framework of the CJS model, it is easier to compare the two models and to understand the limitations of the CJS model. Second, the implementation of our model based on the copula approach is analytically very tractable. In contrast, CGY have to rely on large scale simulation to obtain prices of CDX tranches. Finally, while CGY need information of the term structure of CDX index spreads to fit their model, we fit our model using data at only five year maturity.

The rest of the paper is organized as below. In Section 2, we establish the equivalence between the structural and copula approach to CDO pricing and demonstrate how to implement structural models using the copula approach. Section 3 introduces the data used in our analysis. Section 4 provides detailed analysis on the limitations of the CJS model. Section 5 shows that a double-t or a skewed-t model can jointly price index options and CDX tranches well. Section 6 concludes.
2 Structural vs. Copula Approach: A Synthesis

In this section, we first describe the basic setup of a structural model for credit risk and relate it to the standard copula model. Then we discuss the pricing of CDX index and tranches under our model. Finally, we compare the implementations of our model with that of other structural models in the literature.

2.1 A Structural Model and Relation to the Copula Model

One of the key issues for CDO pricing is how to model default correlation among firms that make up the underlying collateral. In this paper, we take a structural approach to model default correlation by assuming firm asset return satisfies a continuous-time CAPM relation with a systematic and an idiosyncratic factor. Default correlation is due to each firm’s exposure to the same systematic factor.

Specifically, we assume that the log returns of the asset of firm $i$ and the market factor over time interval $[t, t + \tau]$, satisfy the following CAPM specification under the risk-neutral measure $Q$

$$
\ln \left( \frac{A_{i,t+\tau}}{A_{i,t}} \right) = \mu_{i,\tau}(t) + \beta_i \sqrt{\tau} \sigma_M(t) F(t) + \sqrt{\tau} \sigma_{Z_i}(t) Z_i(t),
$$

$$
\ln \left( \frac{M_{t+\tau}}{M_t} \right) = \mu_{M,\tau}(t) + \sqrt{\tau} \sigma_M(t) F(t),
$$

where $\mu_{M,\tau}(t)$ is the expected return of the market and $\mu_{i,\tau}(t)$ is the expected asset return of firm $i$ during the time interval $[t, t + \tau]$. Under the $Q$ measure, both expected returns equal to the risk free rate minus the corresponding dividend rate and the convexity adjustment. $F(t)$ represents the systematic factor, while $Z_i(t)$ represents the idiosyncratic factor. The standardized innovations of the systematic factor $F$ have a cumulative distribution function $P_F$, while the standardized innovations of the idiosyncratic shocks $Z_i$ have a distribution function $P_{Z_i}$, both with zero mean and unit variance. By definition, $F$ is independent of $Z_i$ for all $i$, and $Z_i$ is independent of $Z_k$ for
all $k \neq i$.

Unlike standard CAPM with normally distributed systematic and idiosyncratic factors, we allow $P_F$ and $P_{Z_i}$ to follow fat-tailed distributions with shape parameters $\phi_F(t)$ and $\phi_{Z_i}(t)$, respectively. These shape parameters capture deviations from normality and could measure the skewness, kurtosis, or fatness of the tail for each distribution. For example, if both factors follow the standardized Student-$t$ distribution, then each factor has one shape parameter, the degree of freedom of the Student-$t$ distribution. By allowing fat-tailed distributions in both factors, our model can capture extreme events and improve the modelling of correlated default risk.

Consider a string of time-to-maturity dates $\{\tau_1, \tau_2, \ldots, \tau_j, \ldots, \tau_J\}$. If the current time is $t$, for simplicity, we assume that the firm can default only at $t + \tau_j$, for $j = 1, 2, \ldots, J$. We assume default happens for firm $i$ when the asset value $A_{i, \tau_j}$ is below some exogenously specified default boundary $D_{i, \tau_j}$ at $\tau_j$. Denote the default time of firm $i$ by $\tau^*_i$, then $D_{i, \tau_j}$ is defined such that

$$
\Pr(\tau^*_i \leq \tau_j) = \Pr\left(A_{i, \tau_j} \leq D_{i, \tau_j}\right).
$$

Therefore, the default boundary $D_{i, \tau_j}$ is defined such that

$$
\Pr\left(A_{i, \tau_j} \leq D_{i, \tau_j}\right) \leq \Pr\left(A_{i, \tau_{j+1}} \leq D_{i, \tau_{j+1}}\right).
$$

The number of defaults in the portfolio of $N$ firms by date $\tau_j$ is

$$
\sum_{i=1}^{N} 1\{A_{i, \tau_j} \leq D_{i, \tau_j}\}. \tag{3}
$$

Next we relate our structural model to the standard copula model widely used in industry.

For brevity, we use $\tau$ to represent a generic time-to-maturity date. The total volatility of the asset return of firm $i$ between $t$ and $t + \tau$ is $\sigma_{A_{i, \tau}}(t) = \sqrt{\beta_i^2 \sigma^2_M(t) + \sigma^2_{Z_i}(t)}$. We define $\rho_i(t)$ as the fraction of the total variance of the asset return between $t$ and $\tau$ explained by the systematic factor (i.e., the variance ratio):

$$
\rho_i(t) = \frac{\beta_i^2 \sigma^2_M(t)}{\beta_i^2 \sigma^2_M(t) + \sigma^2_{Z_i}(t)}.
$$

\footnote{We emphasize that we cannot interpret $D_{i, \tau_j}$ as the default boundary in standard structural models, in which default probability is typically calculated as the first passage time to some pre-specified boundary. Instead $D_{i, \tau_j}$ is chosen such that the cumulative default probability of the first passage time equals the probability $\Pr\left(A_{i, \tau_j} \leq D_{i, \tau_j}\right)$.}
where we note the variance ratio does not depend on the time to maturity.

We further define the standardized asset value of firm $i$ (or the Sharpe ratio) as

$$X_{i,\tau}(t) = \frac{\ln \left( \frac{A_{i,t+k}}{A_{i,t}} \right) - \mu_{i,\tau}(t)}{\sigma_{A_{i,\tau}}(t)\sqrt{\tau}},$$

whose cumulative distribution function is $P_{X_{i,\tau}}$.

Based on the above definitions, we have the following alternative representation for $X_{i,\tau}$:

$$X_{i,\tau} = \sqrt{\rho_i} F + \sqrt{1 - \rho_i} Z_i.$$

It is clear that this is a standard copula representation and $X_{i,\tau}$ is independent of time to maturity. We therefore remove the subscript $\tau$. Moreover, $\rho_i$ is the copula correlation, since the correlation between $X_i$ and $X_j$ is $\sqrt{\rho_i \rho_j}$.

Our structural model provides an interesting economic interpretation of the copula model. We show that the copula correlation corresponds to variance ratio, which measures the fraction of the total variance of firm asset return explained by the systematic factor. With this interpretation, our structural model has a simplified representation as the standard copula model. We can draw several important insights from this interpretation. First, the dynamics of the copula correlation are driven by the same factors for the beta of the asset return as well as the systematic and idiosyncratic volatilities. Second, the copula correlation should not vary across tranches as commonly calibrated in copula models. This practice gives rise to the so-called “correlation skew,” referring to the fact the different correlation parameters are needed to fit all the tranches.

The above observation not only provides an economic interpretation to the copula model but also greatly simplifies the implementation of our structural model. While standard implementations of structural models typically rely on data of equity prices of individual firms and market returns to estimate $\{\beta_i, \sigma_M, \sigma_{Z_i}\}$, our model is implemented using the spreads of CDX index and tranches only, similar to how copula models are implemented.
2.2 Marginal Probability of Default and CDX Index Pricing

In this section, we discuss the pricing of CDX index under our structural model. We first compute the marginal default probability at \( \tau \), \( Q_{i,\tau} \), for firm \( i \). Let \( \tau_i^* \) be the default time and the default boundary at \( \tau \) be \( D_{i,\tau} \). Then the probability that firm \( i \) is in default by time \( \tau \) is

\[
Q_{i,\tau}(t) = \Pr(\tau_i^* \leq \tau) = \Pr(A_i, t+\tau \leq D_{i,\tau})
\]

\[
= \Pr\left( X_i(t) \leq \frac{\ln\left( \frac{D_{i,\tau}}{A_{i,t}} \right) - \mu_i,\tau(t)}{\sigma_{A_i,\tau}(t)\sqrt{\tau}} \right)
\]

\[
= P_{\mathcal{X}_i}\left( \frac{\ln\left( \frac{D_{i,\tau}}{A_{i,t}} \right) - \mu_i,\tau(t)}{\sigma_{A_i,\tau}(t)\sqrt{\tau}} \right).
\]

The marginal default probability \( Q_{i,\tau} \) can be computed from CDX spreads and therefore taken as inputs for pricing the CDX tranches. This step simplifies the calibration because we do not need to calibrate the default boundaries and the firm’s total variance, which require additional information from equity data.

Specifically, we can relate the CDX spreads with the marginal default probability. Let \( R \) be the recovery rate and \( d_\tau = \mathbb{E}_t^Q [\exp(-\int_0^\tau r_s ds)] \) be the time-0 default-free discount factor with time-to-maturity \( \tau \). The processes of risk-free rate and default time are assumed to be independent in this paper.

We make the assumption that default occurs only on \( \{\tau_1, \tau_2, ..., \tau_j, ..., \tau_J\} \), and if default happens between \( \tau_{j-1} \) and \( \tau_j \), the recovery value is paid at \( \frac{\tau_{j-1} + \tau_j}{2} \).

The present value of the protection leg of the CDX index at \( t = 0 \) is

\[
CDX_{\text{protection}} = (1 - R) \mathbb{E}_0^Q \left[ \sum_{j=1}^{J} \exp\left( -\int_0^{\frac{\tau_{j-1} + \tau_j}{2}} r_s ds \right) 1_{[\tau^* \in (\tau_{j-1}, \tau_j)]} \right]
\]

\[
= (1 - R) \sum_{j=1}^{J} (Q_{\tau_j} - Q_{\tau_{j-1}}) d_{\tau_j + \tau_{j-1}/2},
\]

where the last inequality is based on the assumption of independence between interest rate and
default risk and the fact that \( E_0^Q \left[ 1_{[\tau^* \in (\tau_{j-1}, \tau_j)]} \right] = Q_{\tau_j} - Q_{\tau_{j-1}} \).

To calculate the present value of the premium leg, we assume that the CDX spread is paid at the end of each protection period if default has not happened by then. The actual payment equals to the CDS spread times the length of the protection period. In case default happens between \( \tau_{j-1} \) and \( \tau_j \), we assume that half of the regular payment is made at \( \frac{\tau_{j-1} + \tau_j}{2} \). Based on these assumptions, the present value of the premium leg for a unit spread is

\[
\begin{align*}
CDX_{\text{premium}} &= E_0^Q \left[ \sum_{j=1}^{J} \exp \left( - \int_0^{\tau_j} r_s ds \right) (\tau_j - \tau_{j-1}) 1_{[\tau^* > \tau_j]} \right] \\
& \quad + E_0^Q \left[ \sum_{j=1}^{J} \exp \left( - \int_0^{\tau_{j-1} + \tau_j} r_s ds \right) \frac{1}{2} (\tau_j - \tau_{j-1}) 1_{[\tau^* \in (\tau_{j-1}, \tau_j)]} \right] \\
& = \sum_{j=1}^{J} (\tau_j - \tau_{j-1}) (1 - Q_{\tau_j}) d_{\tau_j} + \frac{1}{2} \sum_{j=1}^{J} (\tau_j - \tau_{j-1}) (Q_{\tau_j} - Q_{\tau_{j-1}}) d_{(\tau_{j-1} + \tau_{j})/2}.
\end{align*}
\]

Therefore, the CDX spread is simply \( CDX_{\text{protection}}/CDX_{\text{premium}} \). We back out the marginal default probability from CDX spreads and use that as an input for pricing CDX tranches. We emphasize that the above computation is independent of the common factor \( F \).

### 2.3 Conditional Probability of Default and CDO Tranche Pricing

Taking the implied marginal probability of default from the spreads of CDX index as inputs, we turn to the pricing of CDX tranches. The basic challenge is to model default correlation among different firms and to estimate the expected loss from the portfolio of bonds.

Conditional on the systematic factor \( F \), the default probability of firm \( i \) by time \( \tau \) is

\[
Q_{i,\tau}(F) = \Pr(\tau^* \leq \tau \mid F) = \Pr(A_{i,\tau} \leq D_{i,\tau} \mid F) = \Pr \left( Z_i \leq \frac{P_{X_i}^{-1}(Q_{i,\tau}) - \sqrt{\rho_i} F}{\sqrt{1-\rho_i}} \right).
\]

The marginal distribution can be obtained from the conditional distribution from the following
For simplicity we assume that all firms underlying the CDX index are homogeneous in that $\rho_i = \rho$ and $Q_{i,\tau} = Q_\tau$. The computation can be extended to the heterogeneous case as shown in Hull and White (2006).

We first denote $P_{k,\tau}(F)$ as the probability that there are exactly $k$ out of $N$ firms in default by time $\tau$ conditional on the systematic factor $F$. Using conditional independence, we can compute $P_{k,\tau}(F)$ via the binomial formula:

$$P_{k,\tau}(F) = \frac{N!}{(N-k)!k!} Q_\tau(F)^k (1 - Q_\tau(F))^{N-k}. \tag{14}$$

Assuming each firm shares the same constant recovery $R$, the expected loss at time $\tau$, $L_\tau$, is therefore

$$E[L_\tau|F] = \sum_{k=0}^{N} (1 - R) \frac{k}{N} \cdot P_{k,\tau}(F). \tag{15}$$

The unconditional expected loss can be computed by integrating out $F$.

For a tranche $[a, b)$, where $a$ is the attachment point and $b$ is the detachment point, the tranche principle given $k$ defaults is

$$\frac{1}{b - a} \left[ \max \left( b - \frac{k(1-R)}{N}, 0 \right) - \max \left( a - \frac{k(1-R)}{N}, 0 \right) \right],$$

and the expected tranche principle at time $\tau$ is

$$E_\tau(F) = \sum_{k<\frac{aN}{1-R}} P_{k,\tau}(F) + \frac{1}{b - a} \sum_{\frac{aN}{1-R} \leq k < \frac{bN}{1-R}} \left( b - \frac{k(1-R)}{N} \right) P_{k,\tau}(F). \tag{16}$$

The unconditional expectation of the tranche principle, $E_\tau = \int E(F) dF$, can be computed by integrating over the common factor $F$. Numerical integration is needed to compute $E_\tau = \int E(F) dF$. The distribution of the systematic factor for each $\tau$ follows the Student-$t$ distribution
with degree of freedom between 2 and infinity (Gaussian). We use Gaussian-Hermite quadrature with 100 grids in our numerical integration calculation. The loss between $\tau_{j-1}$ and $\tau_j$ is $E_{\tau_{j-1}} - E_{\tau_j}$.

The constraint is that $E_{\tau_{j-1}} - E_{\tau_j} \geq 0$ for all $j$.

When pricing CDX tranches, we make similar assumptions as that for pricing CDX index. Specifically, we assume that between $\tau_{j-1}$ and $\tau_j$, the protection seller will pay to cover the losses in the principle of a given tranche, i.e., $E_{\tau_{j-1}}(F) - E_{\tau_j}(F)$, and the payment is made at $\frac{\tau_{j-1} + \tau_j}{2}$.

Given the expected tranche principle at each point of time, we can compute the expected loss for any time interval and the expected premium, the spread times the remaining principle. The present value of the protection leg is

$$Tranche_{prot} = E^Q_0 \left[ \sum_{j=1}^J \exp \left( - \int_0^{\frac{\tau_{j-1} + \tau_j}{2}} r_s ds \right) \left( E_{\tau_{j-1}}(F) - E_{\tau_j}(F) \right) \right]$$

$$= \sum_{j=1}^J \left[ E_{\tau_{j-1}} - E_{\tau_j} \right] d_{(\tau_j+\tau_{j-1})/2},$$

where the last inequality is based on the assumption of independence between interest rate and default risk and the fact that $E^Q_0 \left[ E_{\tau_{j-1}}(F) - E_{\tau_j}(F) \right] = E_{\tau_{j-1}} - E_{\tau_j}$.

To calculate the present value of the premium leg, we assume that the premium paid is proportional to the outstanding principle before default. For losses between $\tau_{j-1}$ and $\tau_j$, we assume that the premium is proportional to the average principle outstanding and is paid at $\frac{\tau_{j-1} + \tau_j}{2}$. Based on these assumptions, the present value of the premium leg for a unit spread is

$$Tranche_{premium} = E^Q_0 \left[ \sum_{j=1}^J \exp \left( - \int_0^{\tau_j} r_s ds \right) E_{\tau_j}(F) \right]$$

$$+ E^Q_0 \left[ \sum_{j=1}^J \exp \left( - \int_0^{\frac{\tau_{j-1} + \tau_j}{2}} r_s ds \right) \frac{1}{2} \left( E_{\tau_{j-1}}(F) - E_{\tau_j}(F) \right) \right]$$

$$= \sum_{j=1}^J E_{\tau_j} d_{\tau_j} + \frac{1}{2} \sum_{j=1}^J \left[ E_{\tau_{j-1}} - E_{\tau_j} \right] d_{(\tau_j+\tau_{j-1})/2}.$$
spread and up front payment for the tranche from the following formula:

$$O + s \cdot \text{Tranche}_{prem} = \text{Tranche}_{prot}. $$

The term $\text{Tranche}_{prem}$ is also called the “effective duration.” The equity tranche has the shortest duration and the market convention is to pay an up-front payment at the beginning of the contract to compensate for early losses.

3 Data

To have a direct comparison of our model with the CJS model, we use exactly the same data as that used in CJS (2009) and download the data from the website of American Economic Review. The sample period of CJS is between September 22, 2004 and September 19, 2007.

The credit derivatives data used in our analysis are obtained from Markit, which aggregates quotes from different dealers on various credit derivatives and structured products. Our analysis focuses on the data of two types of credit derivatives. The first is the spreads of the DJ CDX North American Investment Grade Index. This index consists of an equally-weighted portfolio of 125 liquid CDS contracts on U.S. companies with corporate debts above investment grade. The running spread on the CDX index can be thought of as the cost of insuring a pre-specified notional amount of an equally-weighted portfolio of risky bonds. Following previous studies, we focus on the CDX index with five year maturity.

The second data we use contain daily spreads on the 0-3, 3-7, 7-10, 10-15, and 15-30 CDX tranches. The CDX tranches are derivative securities with payoff based on the losses on the underlying CDX, and are defined in terms of their loss attachment points. For example, a $1 investment in the 3-7 tranche receives a payoff of $1 if the total losses on the CDX are less than 3%, $0 if total CDX losses exceed 7%, and a payoff that is linearly adjusted for CDX losses.
between 3% and 7%. As with CDS, the tranche prices are quoted in terms of the running spreads that a buyer of protection would have to pay in order to insure the tranche payoff. In the case of tranches, the protection buyer pays the running spread only on the surviving tranche notional on each date. Consequently, if the portfolio losses exceed the tranche’s upper attachment point, the protection buyer ceases to make payments to the protection seller.

In practice, the composition of the CDX index is refreshed every March and September to reflect changes in the composition of the liquid investment grade bond universe. In turn, each new version of the CDX, referenced by a series number, remains on-the-run for six months after the roll date. Since the majority of market activity is concentrated in the on-the-run series, we follow the practice of Longstaff and Rajan (2007) and others to chain the different series to obtain a continuous series of on-the-run spreads during our entire sample period. We also obtain Libor/Swap rates to construct the default-free discount factors.

The index option data of CJS (2009) is obtained from Citigroup, which represent daily over-the-counter quotes on five-year S&P 500 options. These quotes correspond to 13 securities with standardized moneyness levels ranging from 0.70 (30% out-of-the-money) to 1.30 (30% in-the-money) at increments of 5%. Based on these quotes, CJS calibrate the systematic factor of their structural model.

As preliminary analysis, we first back out the implied unconditional default probability each day from the spreads of CDX index assuming a constant 40% recovery rate. In Panel A of Figure 1, we plot the spreads of the CDX index and the implied unconditional default probabilities. The implied default probabilities increase dramatically in May 2005, when GM and Ford were downgraded, and August 2007, when the global financial crisis just began to unfold. Panel B of Figure 1 contains time series plots of the spreads of the five CDX tranches. There is a big discrepancy of the spreads across all the tranches, with the 0-3 equity tranche having the highest
spread and the 15-30 senior tranche the lowest spread. All the spreads increase dramatically with the CDX index in May 2005 and August 2007. All the spreads reach their lowest level in 2006 and early 2007 during the height of the housing bubble.

Panel A of Figure 2 provides time series plots of the Black-Scholes implied volatilities of 30% OTM, ATM, and 30% ITM 5-year index options. The ATM implied volatility fluctuates between 14% and 23% during the sample period. Consistent with the volatility skew in the index option market, OTM implied volatility is higher than ATM implied volatility, which in turn is higher than ITM implied volatility. Panel B of Figure 2 provides time series plots of the ratio between OTM and ATM implied volatility and the ratio between ATM and ITM implied volatility. The ratios measure the steepness of the volatility skew. It is interesting to see that the volatility skew is relatively flat during most part of 2006 and early 2007 and starts to steepen in the third quarter of 2007 when investors start to worry about crash risk.

4 Limitations of the CJS (2009) Model

In this section, we first discuss the CJS model and compares it implementation with that of our model. Then we demonstrate the limitations of the CJS model from three different perspectives. We conclude that the CJS model is not flexible enough to capture the prices of CDX tranches. As a result, the conclusion of CJS (2009) that senior CDX tranches are overpriced is premature.

4.1 The CJS Model

In the CJS model, the return dynamics under the physical measure are given below,

\[
\ln \left( \frac{A_{i,t}}{A_{i,0}} \right) = \mu_{A_{i,t}} \beta_{i} \sigma_{M} \sqrt{\tau} F + \sigma_{Z_{i,t}} \sqrt{\tau} Z_{i},
\]

\[
\ln \left( \frac{M_{t}}{M_{0}} \right) = \mu_{M_{t}} \tau + \sigma_{M} \sqrt{\tau} F.
\]
Compared to our model, the distribution of the systematic factor changes from $F$ (under the $Q$ measure) to $\tilde{F}$ (under the $P$ measure), and the distribution of $Z_i$ is unchanged since idiosyncratic risk is not priced. The expected market return $\mu_{M,\tau} = r_\tau - \delta_\tau + \lambda_\tau - \frac{\sigma^2_{M,\tau}}{2}$, where $\lambda_\tau$ is the equity market risk premium, and firms $i$’s expected asset return $\mu_{A_i,\tau} = r + \lambda_{A_i,\tau} - \frac{\sigma^2_{A_i,\tau}}{2}$, where $\lambda_{A_i,\tau}$ is firm $i$’s risk premium. The CAPM restriction is $\lambda_{A_i,\tau} - \frac{\sigma^2_{A_i,\tau}}{2} = \beta_i(\lambda_\tau - \frac{\sigma^2_{M,\tau}}{2})$.

CJS define the systematic factor as

$$m_\tau = \ln\left(\frac{M_\tau}{M_0}\right) - (r - \delta)\tau = \left(\lambda - \frac{\sigma^2_{M,\tau}}{2}\right)\tau + \sigma_{M,\tau}\sqrt{\tau}\tilde{F} = \sigma_{M,\tau}\sqrt{\tau}F.$$  

They compute $\tilde{Q}_{i,\tau}(m_\tau)$ under the physical measure

$$\tilde{Q}_{i,\tau}(m_\tau) = \Pr(A_{i,\tau} < D_{i,\tau}|m_\tau) = \Pr\left(Z_i \leq \frac{\ln\left(\frac{D_{i,\tau}}{A_{i,0}}\right) - (r\tau + \beta_i m_\tau)}{\sqrt{\sigma_{Z_{i,\tau}}^2/\tau}}\right).$$

And the risk-neutral probability $Q_{i,\tau}$ is computed by integration over $m_\tau$ using the risk-neutral distribution of $m_\tau$ together with estimates of $\left\{\ln\left(\frac{D_{i,\tau}}{A_{i,0}}\right), \beta_i, \sigma_{Z_{i,\tau}}\right\}$. The CDX index spread can be computed using $Q_{i,\tau}$ as shown above. The tranche spreads can be computed similarly.

There are several important differences between our approach and the CJS model. First, while we obtain the estimates of $Q_{i,\tau}$ directly from CDX spreads, CJS need to estimate $\left\{\ln\left(\frac{D_{i,\tau}}{A_{i,0}}\right), \beta_i, \sigma_{Z_{i,\tau}}\right\}$ using CDX spreads and equity market data. Specifically, on each day, CJS pin down the three parameters by matching the CDX index spreads and the average beta and pairwise equity return correlation of the 125 firms. Therefore, the CJS model assumes not only the integration between the index option market and the credit derivatives market, but also the equity and credit derivatives markets.
Second, while CJS assume that the distribution of \( Z_i \) is normal, we allow it to follow Student-\( t \) distribution. As shown later in our empirical analysis, it is very important for the distribution of \( Z_i \) to be fat tailed to accurately capture tranche spreads. This result is also confirmed by CGY (2010).

Finally, while CJS estimate the distribution of \( m_r \) from prices of long-term index options based on a specific functional form of the volatility skew, we allow \( F_r \) to follow Student-\( t \) distribution and estimate the degree of freedom from tranche spreads. The approach of CJS implicitly requires the markets of index options and CDX tranches to be well integrated.

4.2 Limitation 1: Calibration of Beta and Idiosyncratic Volatility

In this section, we show that the way CJS calibrate the beta and volatility of the idiosyncratic factor introduces significant biases in CDX tranche prices. CJS implement their model following the structural approach, which requires information on debt/equity ratio, beta and volatilities of the systematic and idiosyncratic factors. While CJS estimate the distribution of the systematic factor from index option prices, they estimate \( \ln(D_{i,t}, A_{i,t}, 0_i, Z_{i,t}, o) \) on each day by matching the CDX index spreads and the average beta and pairwise equity return correlation of the 125 firms.

In Figure 3, we use the parameters of CJS (2009) to price CDX tranches. Specifically, we use daily parameters of the implied volatility function calibrated from index options in CJS (2009) to obtain the distribution of the systematic factor \( Z_m \) and the volatility parameter \( \sigma_m \). Combining these parameters with daily estimates of \( \left\{ \ln\left(\frac{D_{i,t}}{A_{i,t,0}}\right), \beta_i, \sigma_{Z,i,r}\right\} \) of CJS (2009), we compute the spreads of CDX tranches. Panel A of Figure 3 provides time series plots of the model implied copula correlation, the shape and volatility of the systematic factor. Panel B plots actual spreads and model spreads of all five CDX tranches as well as the RMSE of percentage pricing errors, defined as the difference between market and model price divided by model price. Consistent
with the results of CJS (2009), we find huge pricing errors for all the tranches: The RMSE of percentage pricing errors for five year CDX tranches is 1584.7%! This is one of the main reasons that CJS conclude that senior CDO tranches are overpriced.

In Figure 4, using the same parameters of the systematic factor of CJS, we price CDX tranches based on the copula approach. That is, instead of calibrating \( \ln \left( \frac{D_{i,t}}{A_{i,0}} \right) , \beta_i , \sigma_{Z_{i,t}} \) as CJS, we back out the leverage ratio from CDX index spreads and obtain the copula correlation by fitting the spreads of CDX tranches on a daily basis. The copula correlation in Panel A of Figure 4 looks very different from that in Panel A of Figure 3. More important, we obtain much smaller pricing errors of the CDX tranches: The RMSE of percentage pricing errors is 50%.

For the purpose of pricing CDX tranches, all we need is the variance ratio but not the individual systematic and idiosyncratic volatility. Therefore, the copula approach is more efficient than the structural approach, which has to estimate the individual components of the total variance. Moreover, the pricing of CDX tranches requires the variance ratio under the risk-neutral measure, whereas the estimates in the CJS paper are under the physical measure. Finally, the CJS approach relies on the extra assumption that the equity market is integrated with the credit market.

Therefore, our analysis in this section shows that how to calibrate beta and idiosyncratic volatility can have big impacts on the pricing of CDX tranches. While the CJS approach leads to huge pricing errors for CDX tranches, a simple change of the implementation procedure significantly improves the pricing performance.

### 4.3 Limitation 2: Idiosyncratic Factor Specification

CJS assume that the idiosyncratic factor follows a Gaussian distribution because it represents the conditional asset return given the systematic factor, which follows a skewed distribution. In this section, we show that the Gaussian distribution of the idiosyncratic factor could also introduce
significant biases in CDX tranches prices.

We consider a simple modification of the CJS model by assuming that the idiosyncratic factor follows the Student-\( t \) distribution with a degree of freedom of 3. In Figure 5, using the same parameters of the systematic factor of CJS, we price CDX tranches under this new model based on the copula approach. We obtain the copula correlation by fitting the spreads of CDX tranches on a daily basis. The copula correlation in Panel A of Figure 5 looks very similar to that in Panel A of Figure 4. More important, we obtain much smaller pricing errors for the CDX tranches: The RMSE of percentage pricing errors declines from 50% in Figure 4 to about 33%.

Our result is consistent with that of CGY (2010), who show under a structural model with stochastic volatility and jumps, it is important to introduce jumps in the dynamics of the idiosyncratic factor to price CDX tranches. Therefore, we show that a simple modification of the CJS model by allowing the idiosyncratic factor to follow fat-tailed distribution leads to significantly smaller pricing errors for CDX tranches.

### 4.4 Limitation 3: Systematic Factor Specification

CJS obtain the distribution of the systematic factor under the risk-neutral measure based on the approach of Breeden and Litzenberger (1978). To account for the presence of volatility smile in index options, CJS consider the following hyperbolic tangent function for the implied volatility

\[
\sigma(x, \tau) = a + b \tanh(-c \ln x) \quad (a > b > 0),
\]

where \( x \) denotes moneyness and \( \tau \) is time to maturity. The probability density of the systematic factor is given by the second derivative of the Black-Scholes option pricing formula with respect to the strike price, where the implied volatility follows the above functional form. CJS argue that “within this class of implied volatility functions, we are unable to find a set of option prices that
can jointly price the CDX and CDX tranches...Based on this, we conclude that our model permits only two interpretations of the data: either CDO tranches are mispriced or both index options and corporate bonds are mispriced.”

In Figure 6, we show that the functional form CJS use to back out the distribution of the systematic factor from index options is not flexible enough to capture CDX tranches prices. Specifically, we retain the functional form of the systematic factor in the CJS model, but instead of calibrating the parameters from index option prices, we obtain the parameters by directly fitting the CDX tranches prices. We also calibrate the copula correlation from CDX tranches prices and maintain a Gaussian idiosyncratic factor. Panel A of Figure 6 shows that while the copula correlation resembles that in Figure 5, the shape parameter of the systematic factor looks very different from that of CJS in Figures 3-5. Most important, even though the model is fitted using in sample data, the RMSE of percentage pricing errors is still as high as 45%.

The way CJS price CDX tranches is like an out-of-sample exercise, because they calibrate their model from other markets and use it to price tranches. On the other hand, we price CDX tranches in sample by choosing model parameters to directly fit CDX tranches prices. However, a RMSE of 45% shows that the model is so inflexible that it cannot even fit the tranches in sample. As a result, it is difficult to conclude that based on the poor performance of this model that CDX tranches are mispriced.

Collectively, the above analysis demonstrates from several different perspectives the limitations of the CJS model. It shows that the conclusion of CJS that the CDX tranches are overpriced seems to be premature. Given that the CJS model cannot even capture the CDX tranches prices in sample, we believe it is more reasonable to conclude that the CJS model is misspecified. However, before we can find a model that can jointly price index options and CDX tranches, then there is still the possibility that either the two markets are not well integrated or one of them is not
5 Joint Pricing of Index Options and CDX Tranches

In this section, we develop a new model by simply modifying the distributional assumptions of the systematic and idiosyncratic factors of the CJS model. We show that the model can jointly price index options and CDX tranches very well. One most important feature of our model is that we allow both the systematic and idiosyncratic factors to follow Student-\(t\) distribution in one version of our model. It has been widely recognized that fat tails in both the systematic and idiosyncratic factors are needed to capture CDX tranches prices. This version of the model is simply the double-\(t\) model of Hull and White (2004). Due to the volatility skew for index options, we also consider a skewed-\(t\) model, in which the systematic factor follows a skewed-\(t\) distribution of Hansen (1994) and the idiosyncratic factor follows the Student-\(t\) distribution. While both models have similar pricing errors for CDX tranches, the skewed-\(t\) model has much smaller pricing errors for index options.

5.1 Double-\(t\) Model for CDX Tranches

In this section, we consider the pricing performance of the double-\(t\) model for CDX tranches. The model has three parameters in total: the copula correlation \(\rho \in [0, 1]\) and the inverse of the degree of freedom of Student-\(t\) distribution \(\phi \in \left(0, \frac{1}{2}\right)\) for the systematic and idiosyncratic factors. We apply the following logistic transform to deal with the constraints on parameter values:

\[
\rho = \frac{1}{1 + \exp(\alpha_\rho)},
\]
\[
\phi_F = \frac{1}{2 \left[1 + \exp(\alpha_F)\right]},
\]
\[
\phi_Z = \frac{1}{2 \left[1 + \exp(\alpha_Z)\right]}.
\]
Each day we choose model parameters $\Theta = (\alpha_\rho, \alpha_F, \alpha_Z)$ to minimize the mean squared percentage pricing errors, defined as the difference between market and model spread divided by model spread, of all CDX tranches at five year maturity.\footnote{We use the inverse of the degree of freedom parameters for several reasons. First it transforms an infinite interval to a finite one. Second, it makes the tail distribution somewhat equally sensitive to the entire range of parameter values. For instance, the tail distribution changes little when the degree of freedom goes from 20 to 30, in contrast to the large change in tail from 2 to 3. Using the inverse of the degree of freedom better illustrates the magnitude of the change in the tail distribution. This is especially the case in graphical presentations. Finally, the inverse is positively related to the tail heaviness.}

Panel A of Figure 7 shows that the three estimated parameters $\left(\hat{\rho}, \hat{\phi}_F, \hat{\phi}_Z\right)$ exhibit great variations over time. During the last quarter of 2004 and the first quarter of 2005, the copula correlation is stable within the range of 0.15 and 0.25. During the last quarter of 2006 and the second quarter of 2007, the copula correlation is also very stable around 0.15. During other times, however, the copula correlation tends to change dramatically from a low level of 0.05 to a high level of 0.4. This shows that market perceptions of the correlation in defaults change dramatically over time. The tail parameter ranges from close to zero (which represents a Gaussian distribution) to near 0.5 (which translates to a degree of freedom of 2 for the Student-$t$ distribution, a fat-tailed distribution). The tail parameter of the systematic factor increases steadily from close to zero in the fourth quarter of 2004 to close to 0.5 in the second quarter of 2005. Since then it shows great variation but generally stays above 0.2. This suggests that the level of fat-tailness of the systematic factor has increased overtime and remains volatile. On the other hand, there is a steady decline of the tail parameter of the idiosyncratic factor during the early part of the sample, which then becomes quite volatile during the rest of the sample. The idiosyncratic factor often changes from being to close to Gaussian to very fat-tailed distribution. These results suggest that the Hull and White (2004) model, which sets the degrees of freedom of both the systematic
and idiosyncratic factors to 4, will have serious difficulties in capturing CDX tranches prices.

Panel B of Figure 7 provides time series plots of model implied tranche spreads (up-front for equity tranche) and the actual tranche spreads at the five year maturity, as well as the RMSE of all five year tranches. We see clearly that our model can price all the five year tranches very well. The fit for the senior tranche is especially good. The average RMSE of percentage pricing errors of all the tranches is about 7.53%. This is a dramatic improvement from the CJS model, which has an average RMSE of 1584%!

The double-$t$ model we consider follows almost exactly the same structure as the CJS model. The only changes are the distributions of the systematic and idiosyncratic factors and the way we calibrate the copula correlation. Such simple changes can reduce the pricing errors of CDX tranches from 1584% to 7.53% show that the conclusion of CJS that senior CDX tranches are over priced is really not a robust result.

### 5.2 Double-$t$ model for CDX Tranches and Index Options

While the double-$t$ model can price CDX tranches pretty well, in this section, we test whether this model can simultaneously price both CDX tranches and index options. We repeat the analysis in the previous section. The only difference is each day, we estimate model parameters $\Theta = (\alpha_p, \alpha_F, \alpha_Z)$ by minimizing the mean squared percentage pricing errors of all CDX tranches and index options at five year maturity.

Panel A of Figure 8 reports the three estimated parameters $\left(\hat{\rho}, \hat{\phi}_F, \hat{\phi}_Z\right)$ as well as the volatility of the systematic factor $\hat{\sigma}_M$. While the parameters exhibit similar patterns as that in Panel A of Figure 7, all of them become less volatile and exhibit less extreme fluctuations. It could be that the options prices provide additional information to better identify these parameters. The systematic volatility is quite stable during the entire sample except for a couple of episodes during
the second and third quarters of 2005 and the third quarter of 2007.

Panel B of Figure 8 provides time series plots of model implied tranche spreads (up-front for equity tranche) and the actual tranche spreads at the five year maturity, as well as the RMSE of all five year tranches. Despite the differences in model parameters shown in Panel A of Figure 8, our model can still price all the five year tranches very well. The fit for the senior tranche is again quite good. The average RMSE of percentage pricing errors of all the tranches increases slightly to 7.67%.

Panel C of Figure 8 shows that our model can price index options pretty well too. The average RMSE of percentage pricing errors is about 7.81%. However, the double-\(t\) model cannot perfectly capture the implied volatility skew. For example, the model tends to overprice ATM options, while underprice deep OTM options. This implies that the implied volatility skew predicted by the model is flatter than that is observed in the data. One main reason is that the Student-\(t\) distribution is symmetric. To capture the volatility skew, a skewed distribution is needed.

5.3 Skewed-\(t\) model for CDX Tranches and Index Options

To better capture the price of index options, in this section, we introduce a skewed-\(t\) model in which the systematic factor follows the skewed-\(t\) distribution of Hansen (1994) and the idiosyncratic factor still follows the standard Student-\(t\) distribution.

The skewed-\(t\) distribution of Hansen (1994) with parameters \((\nu, \lambda)\) for degree of freedom and skewness, respectively, has the following density function

\[
g(z|\nu, \lambda) = \begin{cases} 
bc \left(1 + \frac{1}{\nu-2} \left(\frac{b z + a}{1-\lambda}\right)^2\right)^{-\left(\nu+1\right)/2}, & z < -a/b \\
bc \left(1 + \frac{1}{\nu-2} \left(\frac{b z + a}{1-\lambda}\right)^2\right)^{-\left(\nu+1\right)/2}, & z \geq -a/b
\end{cases}
\]

where \(2 < \nu < \infty\), \(-1 < \lambda < 1\), \(a = 4\lambda c \left(\frac{\nu-2}{\nu-1}\right)\), \(b^2 = 1 + 3\lambda^2 - a^2\), and \(c = \frac{\Gamma \left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu-2)}\Gamma \left(\frac{\nu}{2}\right)}\). We choose \(\lambda = -0.5\) to obtain a negatively skewed systematic factor and estimate the degree of
freedom from the prices of both CDX tranches and index options.

Panel A of Figure 9 reports the three estimated parameters \( \hat{\rho}, \hat{\phi}_F, \hat{\phi}_Z \) as well as the volatility of the systematic factor \( \hat{\sigma}_M \). One big difference from the results in Figure 8 is that the copula correlation becomes much more stable. It could be that the negatively skewed systematic factor generates higher correlated default risk. As a result, the copula correlation does not have to change dramatically to price CDX tranches.

Panel B of Figure 9 provides time series plots of model implied tranche spreads (up-front for equity tranche) and the actual tranche spreads at the five year maturity, as well as the RMSE of all five year tranches. The skewed-\( t \) model has almost the same performance as the double-\( t \) model in pricing all five year CDX tranches. The average RMSE of percentage pricing errors all the tranches increases slightly to 7.85%.

Panel C of Figure 9 shows that the skewed-\( t \) model can price index options extremely well. The average RMSE of percentage pricing errors decreases dramatically from 7.81% for the double-\( t \) model to about 0.96% for the skewed-\( t \) model. The model does not exhibit obvious pricing biases for ATM and OTM options and therefore can capture the volatility skew very well.

We emphasize that we have not searched hard for some exotic models that are able to price both CDX tranches and index options. In fact, the skewed-\( t \) distribution is one of many probability distributions that have fat tails and negative skewness. Other models can probably have equal or better pricing performance than the skewed-\( t \) model. The larger point we want to make is that if simple modifications to the CJS model can lead to such dramatic improvements and excellent results in pricing CDX tranches and index options, it is really difficult to conclude with any confidence that these contracts are not efficiently priced.
6 Conclusion

In this paper, we revisit an important issue raised by the influential study of Coval, Jurek and Stafford (2009). Using a Merton structural model within a CAPM framework, CJS argue that senior CDX tranches are overpriced relative to S&P 500 index options because their model cannot reconcile the prices of these two markets. We carefully examine the limitations of the CJS model and its implementations from several different perspectives and provide compelling evidence that the CJS model is not flexible enough to capture CDX tranches prices. On the other hand, we develop a simple model that can simultaneously price CDX tranches and index options. Our results show that the conclusion of CJS that CDX tranches are overpriced is premature and that the tranches market is actually efficient.
REFERENCES

L. Andersen and J. Sidenius. Extensions to the gaussian copula: random recovery and random

Bekaert, Geert and Engstrom, Eric C., Asset Return Dynamics Under Bad Environment Good


F. Black and M. Scholes. The pricing of options and corporate liabilities. Journal of Political

C. Bluhm and O. Ludger, 2007, Structured Credit, Portfolio Analysis, Baskets & CDOs, Chap-
man and Hall.

D. T. Breeden and Litzenberger. Prices of stochastic contingent claims implicit in option prices.

P. Collin-Dufresne, R. Goldstein, and F. Yang, 2010, On the relative pricing of long maturity


D. Duffie and N. Garleanu. Risk and valuation of collateralized debt obligations. Financial

D. Duffie, L. Saita, and K. Wang. Multi-period corporate default prediction with stochastic


Figure 1: Spreads and Implied Hazard Rates of CDX Index and Prices of CDX Tranches.
This figure plots the running spread and implied hazard rate of the CDX NA IG index at five year maturity. It also plots the prices of the 0-3, 3-7, 7-10, 10-15, and 15-30 CDX tranches at five year maturity. The sample period is between September 22, 2004 and September 19, 2007.

Panel A: CDX

Panel B: Tranches
Figure 2: Five-Year SPX Options.  
This figure plots the level and slope of Black-Scholes implied volatility of the five-year SPX options. The sample period is between September 22, 2004 and September 19, 2007.

Panel A: BS Implied Vol.

Panel B: Slope of BS Implied Vol.
OTM: Strike/spot = 0.7; ITM: Strike/spot = 1.3.
Figure 3: Parameters Estimates and Pricing Performance from CJS (2009) Model.
We provide time series plots of daily estimates of model parameters of the CJS-Normal model using CDX tranches spreads, and the model implied tranches spreads.

Panel A: Model Parameter Estimates
The model has two parameters, the copula correlation (between zero and one), and the shape parameter of the systematic factor, both fixed at the estimates from the CJS paper. A higher value of correlation parameter suggests a higher loading on the systematic factor and a higher value for systematic factor suggests a heavier downside tail.
Panel B: Pricing Performance
This figure plots the implied tranches spreads (up-front for the equity tranche) from the model and the actual tranche spreads for all the standardized CDX tranches during our sample, as well as the daily RMSE (pricing errors defined as (data-model)/model) of the five tranches. Over the sample period, the average RMSE for five year CDX tranches is 1584.1%.
We provide time series plots of daily estimates of model parameters of the CJS-Normal model using CDX tranche spreads, and the model implied tranche spreads. The model is estimated by minimizing the percentage pricing errors of the tranche spreads (up-front for equity tranche), defined as $(\text{data-model})/\text{model}$.

Panel A: Model Parameter Estimates

The model has two parameters, the copula correlation (between zero and one), estimated from tranche spreads, and the shape parameter of the systematic factor, fixed at the estimates from the CJS paper. A higher value of correlation parameter suggests a higher loading on the systematic factor and a higher value for systematic factor suggests a heavier downside tail.
Panel B: Pricing Performance
This figure plots the implied tranches spreads (up-front for the equity tranche) from the model and the actual tranche spreads for all the standardized CDX tranches during our sample, as well as the daily RMSE (pricing errors defined as (data-model)/model) of the five tranches. Over the sample period, the average RMSE for five year CDX tranches is 50.07%.
Figure 5: Alternative Distribution Assumption for the Idiosyncratic Factor within the CJS (2009) Model - Parameter Estimates and Pricing Performance of the CJS-T(3) Copula Model.

We change the distribution assumption for the idiosyncratic factor in the CJS model from normal to student's t with degree of freedom 3, which has a heavy tail distribution. We provide time series plots of daily estimates of model parameters of the CJS-T(3) model using CDX tranches spreads, and the model implied tranches spreads. The model is estimated by minimizing the percentage pricing errors of the tranches spreads (up-front for equity tranche), defined as (data-model)/model.

Panel A: Model Parameter Estimates
The model has two parameters, the copula correlation (between zero and one), estimated from tranches spreads, and the shape parameter of the systematic factor, fixed at the estimates from the CJS paper. A higher value of correlation parameter suggests a higher loading on the systematic factor and a higher value for systematic factor suggests a heavier downside tail.

![Copula Correlation](image1)

![Systematic Factor (CJS Estimates)](image2)
Panel B: Pricing Performance
This figure plots the implied tranches spreads (up-front for the equity tranche) from the model and the actual tranche spreads for all the standardized CDX tranches during our sample, as well as the daily RMSE (pricing errors defined as (data-model)/model) of the five tranches. Over the sample period, the average RMSE for five year CDX tranches is 33.05%.
Figure 6: Estimation of Copula Correlation and Systematic Factor Parameters Based on CDX Tranches Spreads within the CJS (2009) Model - Parameter Estimates and Pricing Performance. We provide time series plots of daily estimates of model parameters of the CJS-Normal model using CDX tranches spreads, and the model implied tranches spreads. The model is estimated by minimizing the percentage pricing errors of the tranches spreads (up-front for equity tranche), defined as (data-model)/model.

Panel A: Model Parameter Estimates
The model has two parameters, the copula correlation (between zero and one), estimated from tranches spreads, and the shape parameter of the systematic factor, fixed at the estimates from the CJS paper. A higher value of correlation parameter suggests a higher loading on the systematic factor and a higher value for systematic factor suggests a heavier downside tail.
Panel B: Pricing Performance

This figure plots the implied tranche spreads (up-front for the equity tranche) from the model and the actual tranche spreads for all the standardized CDX tranches during our sample, as well as the daily RMSE (pricing errors defined as (data-model)/model) of the five tranches. Over the sample period, the average RMSE for five year CDX tranches is 45.32%.
Figure 7: Parameter Estimates and Pricing Performance of the Double-t Copula Model Based on CDX Tranches Spreads.

We change the distribution assumptions for the systematic and idiosyncratic factor in the CJS model student's t distributions with the degree of freedom parameters estimated from the data. We provide time series plots of daily estimates of model parameters of the Double-t model using CDX tranches spreads, and the model implied tranches spreads. The model is estimated by minimizing the percentage pricing errors of the tranches spreads (up-front for equity tranche), defined as (data-model)/model.

Panel A: Model Parameter Estimates

The model has three parameters, the copula correlation (between zero and one) and the inverse of the degrees of freedom of the systematic and idiosyncratic factors (between zero and 0.5). A higher value of correlation parameter suggests a higher loading on the systematic factor and a higher value for systematic/idiosyncratic factor suggests a heavier downside tail.
Panel B: Pricing Performance

This figure plots the implied tranches spreads (up-front for the equity tranche) from the model and the actual tranche spreads for all the standardized CDX tranches during our sample, as well as the daily RMSE (pricing errors defined as (data-model)/model) of the five tranches. Over the sample period, the average RMSE for five year CDX tranches is 7.53%.
Figure 8: Joint Pricing of CDX Tranches Spreads and OTM Index Put Options - Parameter Estimates and Pricing Performance of the Double-t Copula Model.

We change the distribution assumptions for the systematic and idiosyncratic factor in the CJS model student's t distributions with the degree of freedom parameters estimated from the data. We provide time series plots of daily estimates of model parameters of the Double-t model using CDX tranches spreads and Index options prices, and the model implied tranches spreads and options prices. The model is estimated by minimizing the percentage pricing errors of the tranches spreads (up-front for equity tranche) and options prices, defined as (data-model)/model.

Panel A: Model Parameter Estimates
The model has three parameters, the copula correlation (between zero and one) and the inverse of the degrees of freedom of the systematic and idiosyncratic factors (between zero and 0.5). A higher value of correlation parameter suggests a higher loading on the systematic factor and a higher value for systematic/idiosyncratic factor suggests a heavier downside tail.
Panel B: Pricing Performance of CDX tranches
This figure plots the implied tranches spreads (up-front for the equity tranche) from the model and the actual tranche spreads for all the standardized CDX tranches during our sample, as well as the daily RMSE (pricing errors defined as (data-model)/model) of the five tranches. Over the sample period, the average RMSE for five year CDX tranches is 7.67%. 
Panel C: Pricing Performance of OTM Put Options Prices
This figure plots the implied options prices from the model and the actual prices for all the OTM put options during our sample, as well as the daily RMSE (pricing errors defined as \((\text{data-model})/\text{model}\)) of the five tranches. Over the sample period, the average RMSE for five year options prices is 7.81%. 
Figure 9: Joint Pricing of CDX Tranches Spreads and OTM Index Put Options - Parameter Estimates and Pricing Performance of the Skewed-t Copula Model.

We change the distribution assumptions for the systematic and idiosyncratic factor in the CJS model to skewed-t and student's t distributions with the degree of freedom parameters estimated from the data. The skewed-t distribution follows Hansen (1994) with the skewness parameter set at -0.5, a negatively skewed systematic factor. We provide time series plots of daily estimates of model parameters of the Double-t model using CDX tranches spreads and Index options prices, and the model implied tranches spreads and options prices. The model is estimated by minimizing the percentage pricing errors of the tranches spreads (up-front for equity tranche) and options prices, defined as (data-model)/model.

Panel A: Model Parameter Estimates
The model has three parameters, the copula correlation (between zero and one) and the inverse of the degrees of freedom of the systematic and idiosyncratic factors (between zero and 0.5). A higher value of correlation parameter suggests a higher loading on the systematic factor and a higher value for systematic/idiosyncratic factor suggests a heavier downside tail.
Panel B: Pricing Performance of CDX tranches
This figure plots the implied tranches spreads (up-front for the equity tranche) from the model and the actual tranche spreads for all the standardized CDX tranches during our sample, as well as the daily RMSE (pricing errors defined as (data-model)/model) of the five tranches. Over the sample period, the average RMSE for five year CDX tranches is 7.85%.
Panel C: Pricing Performance of OTM Put Options Prices

This figure plots the implied options prices from the model and the actual prices for all the OTM put options during our sample, as well as the daily RMSE (pricing errors defined as (data-model)/model) of the five tranches. Over the sample period, the average RMSE for five year options prices is 0.96%.