Banking Bubbles and Financial Crisis*

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Abstract

This paper develops a macroeconomic model with a banking sector in which banks face endogenous borrowing constraints. There is no uncertainty about economic fundamentals. Banking bubbles can emerge through a positive feedback loop mechanism. Changes in household confidence can cause the collapse of bubbles, resulting in a financial crisis. Credit policy can mitigate economic downturns but also incur an efficiency loss. Bank capital requirements can prevent the formation of banking bubbles by limiting leverage. But a too restrictive requirement leads to less lending and hence less production.

Keywords: Banking Bubble, Multiple Equilibria, Financial Crisis, Self-fulfilling Prophecy, Credit Policy, Capital Requirements, Borrowing Constraints
JEL codes: E2, E44, G01, G20

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1 Introduction

The recent financial crisis resulted in the collapse of large financial institutions, the bailout of banks by national governments, and downturns in stock markets around the world. In addition, it contributed to persistent high unemployment, failures of key businesses, declines in consumer wealth, real investment, and output. The financial crisis was caused by a complex interplay of valuation and liquidity problems in the United States banking system in 2007. The bursting of the U.S. housing bubble, which peaked in 2007, caused the values of securities tied to U.S. real estate pricing to plummet, damaging financial institutions globally. Concerns about bank solvency, declines in credit availability, and damaged investor confidence caused the global stock markets to fall. Declines in bank lending reduced real investment and output, causing the Great Recession.

The recent financial crisis provides a challenge to macroeconomists. Traditional macroeconomic models typically assume perfect financial markets and ignore financial frictions. These models are not useful for understanding financial crisis. Bernanke and Gertler (1989), Carlstrom and Fuerst (1997), Kiyotaki and Moore (1997), and Bernanke, Gertler and Gilchrist (1999) introduce financial frictions into business cycle models. These models assume that financial frictions appear only in non-financial firms and treat financial intermediaries as a veil. These models cannot capture the fact that the current financial crisis featured a significant disruption of financial intermediation.

In this paper, we develop a macroeconomic model with a banking sector in which changes in household confidence can cause a financial crisis. To focus on the impact of household confidence, we assume that there is no uncertainty about economic fundamentals. The key idea of the model is to introduce financial frictions into the banking sector in the form of endogenous borrowing constraint similar to those in Albuquerque and Hopenhayn (2004), Alvarez and Jermann (2000), Jermann and Quadrini (2011), and Miao and Wang (2011a,b,c). In our model, households put deposits in a bank and deposits become liabilities of the bank. The bank has limited commitment and may default on deposit liabilities. If the bank chooses to default, then depositors can seize a fraction of bank capital. Instead of liquidating seized bank capital, depositors reorganize the bank and keep it running. The bank and depositors renegotiate deposit repayments. The threat value to depositors is the stock market value of the bank with seized bank capital. Suppose that the bank has a full bargaining power. Then deposits cannot exceed the threat value to depositors or the stock market value of the reorganized bank. This constraint is incentive compatible for both depositors and the bank. It also ensures that there is no default in an optimal contract.

We show that a banking bubble can exist given the endogenous borrowing constraint. The intuition is based on the following positive feedback loop mechanism: If both depositors and the bank have optimistic beliefs about the bank value, then the bank wants to take more deposits and households are willing to put more deposits in the bank, in the hope that deposit repayments can be backed by a high bank value. The bank then uses the increased deposits to make more lending to non-financial firms. Consequently, the bank can make more profits, which makes the bank value indeed high, justifying the initial optimistic beliefs. We call this equilibrium bubbly equilibrium.

Of course, there is another equilibrium, called bubbleless equilibrium, in which no one believes in bubbles. Then households place less deposits in the bank because they are concerned that their deposits cannot be repaid in the future. In this case, banks make less lending to non-financial firms, resulting in lower capital stock and lower output.

As in Blanchard and Watson (1982), Weil (1987), Kocherlakota (2009), and Miao and Wang (2011a), we construct a third type of equilibrium in which households believe that banking bubbles may burst in the future with some probability. We show that even though there is no shock to the fundamentals of the economy, changes in confidence trigger a financial crisis. We show that immediately following the collapse of the banking bubble, deposits shrink, lending falls, and credit spreads rise, causing real investment and output to fall.

During the recent financial crisis, the Federal Reserve conducted three general types of credit policy. The first is discount window lending. The Fed used the discount window to lend funds to commercial banks that in turn lent them out to non-financial borrowers. The second is direct lending. The Fed lent directly in high grade credit markets, funding assets that included commercial paper, agency debt and mortgage backed securities. The third is equity injections. The Treasury coordinated with the Fed to acquire ownership positions in commercial banks by injecting equity.

To assess the impact of credit policy, we introduce a central bank in our model. we allow the central bank to act as intermediary by borrowing funds from savers and then lending them to investors. Unlike private intermediaries, the central bank does not face constraints on its leverage ratio. There is no enforcement problem between the central bank and its creditors because it can commit to always honoring its debt. For simplicity, we assume that the central bank's direct lending is financed by raising lump-sum taxes. On the other hand, the central bank is less efficient in intermediating funds than commercial banks. Following Gertler and Karadi (2011) and Gertler and Kiyotaki (2010), we model this inefficiency as a deadweight loss of output. We also follow their studies and assume that the size of direct lending responds to credit spreads according to a feedback rule. In our model, credit spreads rise sharply at the onset of a crisis. The central bank then injects credit in response to movements in credit spreads, according to the feedback rule. We show that this credit policy can mitigate economic downturns. The net effect on welfare trades off between this benefit and efficiency costs.

We also use our model to study the role of capital requirements. Bank capital requirements ensure that banks are not participating or holding investments that increase the risk of default and that they have enough capital to sustain operating losses while still honoring deposit withdrawals. In our model, there is no uncertainty about fundaments and hence there is no issue of risk-taking behavior. However, bank capital requirements can still help stabilize the banking system. We show that these requirements can help prevent the formation of a banking bubble. The intuition is that capital requirements limit leverage. When these requirements are sufficiently restrictive, banks cannot borrow excessively from households and hence the positive feedback loop discussed earlier cannot be initiated. Limiting leverage, however, comes at a cost because it will reduce lending to non-financial firms. As a result, it will reduce investment and output.

Our paper is related to three strands of literature. First, it is related to the recent literature that incorporates a financial sector into macroeconomic models (e.g., Gertler and Karadi (2011), Gertler and Kiyotaki (2010), and Gertler, Kiyotaki, and Queralto (2011), and Brunnermeier and Sannikov (2011)). For the financial sector to play an important role, one needs to introduce various frictions explicitly in the financial sector so that the Modigliani and Miller theorem does not hold. Frictions are typically modeled in the form of borrowing constraints. Borrowing constraints can be micro-founded by agency issues or moral hazard problems. Our paper differs from this literature in two respects. First, there is no uncertainty about economic fundamentals in our model. Unlike the studies cited earlier, which assumes that financial crisis is triggered by some exogenous shocks (e.g., capital quality shocks), we show that financial crisis is triggered by changes in agents' beliefs about the stock market value of banks. Second, we introduce borrowing constraints from an optimal contracting problem with limited commitment, rather than from agency issues or moral hazard problems. This contracting problem is tractable to analyze in our deterministic setup.

Our model is also related to the literature on rational bubbles.[1] It is well known that it is nontrivial to generate rational bubbles in infinite-horizon models (Santos and Woodford (1997)). Rational bubbles are often studied in overlapping-generations models (e.g., Tirole (1985) and Martin and Ventura (2011)). Rational bubbles can also be generated in infinite-horizon models with borrowing constraints (e.g., Kocherlakota (1992, 2009), Hirano and Yanagawa (2011), and Wang and Yi (2011)). One limitation of all these models is that they study bubbles on intrinsically useless assets or on assets with exogenously given payoffs. Miao and Wang (2011a) provide a theory of credit-driven stock price bubbles in an infinite-horizon model with production. Stock dividends are endogenously affected by bubbles. Miao and Wang (2011b,c) apply this theory to study endogenous total factor productivity and endogenous growth, respectively. The present paper also borrows ideas from Miao and Wang (2011a) in that banking bubbles in this paper are created by a positive feedback loop mechanism just as in Miao and Wang (2011a).

Finally, our paper is related to the literature on bank runs (e.g., Diamond and Dybvig (1983)). A bank run occurs when a large number of bank customers withdraw their deposits because they believe the bank might fail. To sustain a bank run equilibrium, the following positive feedback loop mechanism must be at work: As more people withdraw their deposits, the likelihood of default increases, and this encourages further withdrawals. Panic by people can cause a sound bank to fail. However, if all people believe that the bank is sound, then no large withdrawals happen and hence a bank run does not occur. Both types of equilibria are self-fulfilling. This literature typically considers an essentially static setup (or a three-period setup) without explicit dynamics. He and Xiong (2011) develop a dynamic model of debt run and derive a unique equilibrium. Our paper differs from this literature in that our model features both the financial and non-financial sectors in a dynamic macroeconomic model. We focus on the question of how bubbles and crashes in banking stocks affect financial crisis and the real economy.

The remainder of the paper proceeds as follows. Section 2 presents a baseline model. Section 3 provides equilibrium characterizations. Section 4 studies equilibrium with stochastic bubbles. Section 5 analyzes the role of bank capital requirements. Section 6 studies the impact of credit policy. Section 7 concludes. An appendix collects all technical proofs.

[1] See Scheinkman and Xiong (2003) for a model of bubbles based on heterogeneous beliefs. See Shiller (2005) for a discussion of bubbles based on irrational exuberance. See Brunnermeier (2009) for a survey of various theories of bubbles.

2 A Baseline Model

We start with a baseline model with deterministic bubbles. Consider a deterministic economy consisting of households, non-financial firms and financial intermediaries (or simply banks). We do not consider government or monetary authority for now. Time is continuous and continues forever.

2.1 Households

Following Gertler and Karadi (2011) and Gertler and Kiyotaki (2010), we formulate the household sector in a way that maintains the tractability of the representative agent approach. There is a continuum of identical households of measure unity. Each household consumes, saves and supplies labor. Normalize labor supply to unity. Households save by lending funds to competitive banks. Within each household, there are two types of members: workers and bankers. Workers supply labor and return their wages to the household. Each banker manages a bank and transfers dividends back to the household. The household owns the bank and deposits funds in banks. (These banks may be owned by other households.) Within the family, there is perfect consumption insurance.

Households do not hold capital directly, but own all non-financial firms by trading firm shares in a stock market. Assume that households lend funds to banks that they do not own. Each household derives utility from consumption $\{C_t\}$ according to the linear utility, $\int_0^\infty e^{-rt}C_t dt$, where r is the subjective discount rate. Because of linear utility, r is also equal to the interest rate on riskless bonds. The budget constraint is given by:

$$dD_t = rD_t dt - C_t dt + w_t dt + \Pi_t dt, \tag{1}$$

where D_t, w_t, and Π_t represent deposits, the wage rate, and dividends from both financial and non-financial firms, respectively.

2.2 Banks

There is a continuum of banks of measure unity. Banks lend funds obtained from households to non-financial firms. Banks are identical and there is no liquidity risk and no interbank market. Thus, we only need to consider a representative bank's behavior. At each time t, let N_t be the net worth that a bank has, D_t the deposits raised from households, and S_t the loans lent to non-financial firms. Non-financial firms use loans to finance capital expenditures K_t. We consider financial frictions in the banking sector only. We assume that there is no financial or real friction in the non-financial firm sector, and no friction in transferring funds between a bank and non-financial firms. Thus, the price of capital is equal to unity. The bank's balance sheet satisfies:

$$N_t + D_t = S_t = K_t. \tag{2}$$

The banker maximizes the stock market value of the bank at any time t, denoted by $V_t(N_t)$. Note that we have suppressed the aggregate state variables as arguments in the value function V_t. This value function satisfies the Bellman equation:

$$V_t(N_t) = \max_{C_t^b, D_t} \int_t^T e^{-rs} C_s^b ds + e^{-r(T-t)} V_T(N_T), \text{ any } T > t, \tag{3}$$

subject to some constraints to be specified next. Here $\{C_t^b\}$ represents bank dividends. The first constraint is the flow of funds constraint given by:

$$dN_t = r_{kt} N_t dt + (r_{kt} - r) D_t dt - C_t^b dt, \tag{4}$$

where r_{kt} represent the lending rate and r is the deposit rate. As long as the lending rate r_{kt} is higher than the deposit rate r, the bank prefers to keep accumulating assets until it overcomes financial frictions in the form of borrowing constraints. In the literature, there are several modeling strategies to limit bankers' ability to save to overcome borrowing constraints. For example, Gertler and Karadi (2011) and Gertler and Kiyotaki (2010) assume that a banker exits next period with a constant probability and becomes a worker. His position is then replaced by a randomly selected worker, keeping the number in each occupation constant. Brunnermeier and Sannikov (2011) assume that bankers may choose to retire.

In this paper, we assume that the banker must pay out dividends as a fraction $\theta \in (0, 1)$ of his net worth:

$$C_t^b \geq \theta N_t. \tag{5}$$

The motivation for such constraint requires a richer micro-founded model that may involve a combination of asymmetric information and a divergence of interests between shareholders and managers. For our purpose, one may view (5) as a modeling shortcut for forcing banks to pay out dividends, instead of keeping accumulating assets. Using the approach of Gertler and Karadi (2011) or Gertler and Kiyotaki (2010) will also work and will not change our key insights.

The second constraint is that deposits and net worth cannot be negative, because the bank can only borrow from the households:

$$D_t \geq 0, \;\; N_t \geq 0. \tag{6}$$

The third constraint is a borrowing constraint. As long as the lending rate r_{kt} is higher than the deposit rate r, the bank prefers to expand its assets as much as possible by borrowing additional funds from households. To limit this ability, we introduce the following borrowing constraint:

$$D_t \leq V_t\left(\xi N_t\right), \tag{7}$$

where $\xi \in (0, 1]$ represents the degree of financial frictions. This is the key innovation of the model, which requires some explanations. The micro-foundation of this borrowing constraint is based on the optimal contract between the bank and households (depositors) with limited commitment/enforcement. To best understand this, we consider a discretized setup. Suppose time is denoted by $t = 0$, dt, $2dt$, $3dt, ...$ At each time t, the contract specifies a deposit D_t to the bank and a repayment $e^{rdt} D_t$ to the households at time $t + dt$. The bank lends out $N_t + D_t$ to non-financial firms and earn returns $e^{r_{kt}dt}\left(N_t + D_t\right)$. It pays out dividends $C_t^b dt$. Thus its flow of funds constraint is given by:

$$N_{t+dt} = e^{r_{kt}dt} N_t + \left(e^{r_{kt}dt} - e^{rdt}\right) D_t - C_t^b dt.$$

If the bank decides to repay deposits to the households, its value is given by:

$$\begin{aligned} & C_t^b dt + e^{-rdt} V_{t+dt}\left(N_{t+dt}\right) \\ = \quad & e^{r_{kt}dt} N_t + \left(e^{r_{kt}dt} - e^{rdt}\right) D_t - N_{t+dt} + e^{-rdt} V_{t+dt}\left(N_{t+dt}\right), \end{aligned}$$

where we have used the above flow of funds constraint to substitute out C_t^b. However, the bank has limited commitment. It may take deposits D_t and default on the deposit liabilities by not repaying $e^{rdt} D_t$. If this happens, the depositors can capture a fraction ξ of bank net worth (or bank capital) N_t. Instead of liquidating these assets, the depositors reorganize the bank and keep it running in the next period using recovered bank capital ξN_t. The bank and depositors renegotiate deposit repayments by Nash bargaining. Suppose the bank has all bargaining power. The depositors can only get the threat value, which is the stock market value of the reorganized bank $e^{-rdt} V_{t+dt}\left(\xi N_t\right)$. The bank gets the remaining value.

Enforcement of the contract requires the value of not defaulting is not smaller than the value of defaulting, i.e.,

$$\begin{aligned} & e^{r_{kt}dt}N_t + \left(e^{r_{kt}dt} - e^{rdt}\right) D_t - N_{t+dt} + e^{-rdt}V_{t+dt}\left(N_{t+dt}\right) \\ \geq \; & e^{r_{kt}dt}N_t + e^{r_{kt}dt}D_t - N_{t+dt} + e^{-rdt}V_{t+dt}\left(N_{t+dt}\right) - e^{-rdt}V_{t+dt}\left(\xi N_t\right). \end{aligned}$$

This incentive constraint ensures that there is no default in an optimal contract. Simplifying this constraint yields:

$$e^{rdt}D_t \leq e^{-rdt}V_{t+dt}\left(\xi N_t\right).$$

Taking the continuous time limits as $dt \to 0$ yields (7).

Note that the borrowing constraint in (7) is different from that in Gertler and Karadi (2011) or Gertler and Kiyotaki (2010). These studies assume that bankers can divert a fraction λ of assets S_t. If this happens, the depositors force the closure of the bank and recover the remaining fraction of assets, $(1-\lambda) S_t$. In this case, the incentive constraint becomes:

$$V_t\left(N_t\right) \geq \lambda S_t.$$

2.3 Non-financial Firms

There is a continuum of identical non-financial firms of measure unity. Each non-financial firm produces output using a constant-returns-to-scale technology with capital and labor inputs. We may write aggregate production function as:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}, \quad \alpha \in (0,1),$$

where Y_t, K_t, and L_t represent aggregate output, capital and labor respectively. Competitive profit maximization implies:

$$(1-\alpha) K_t^{\alpha} L_t^{-\alpha} = w_t.$$

It follows that the gross profits per unit of capital are given by:

$$\frac{Y_t - w_t L_t}{K_t} = \alpha K_t^{\alpha-1} L_t^{1-\alpha}.$$

Assume that there is no real or financial friction in the non-financial firm sector. A non-financial firm obtains funds from banks by issuing equity at the price of unity. The firm uses the funds to purchase capital K_t and returns dividends $\alpha K_t^{\alpha-1} L_t^{1-\alpha}$ to the banks. Thus, the lending rate is equal to the capital (or equity) return:

$$r_{kt} = \alpha K_t^{\alpha-1} L_t^{1-\alpha} - \delta, \tag{8}$$

where δ is the depreciation rate of capital. Equation (8) also implies that the marginal product of capital $\alpha K_t^{\alpha-1} L_t^{1-\alpha}$ is equal to the user cost of capital, which is equal to the sum of the lending rate r_{kt} and the depreciation rate δ.

2.4 Competitive Equilibrium

A competitive equilibrium consists of quantities $\{C_t, C_t^b, K_t, N_t, D_t, L_t\}$, prices $\{r_{kt}, w_t\}$, and the value function $\{V_t(N_t)\}$ such that households, bankers and firms optimize and markets clear: $L_t = 1$ and

$$dK_t = K_t^{\alpha} L_t^{1-\alpha} dt - \delta K_t dt - C_t dt. \tag{9}$$

3 Equilibrium Characterizations

In this section, we first analyze a single bank's decision problem taking prices $\{r_{kt}, w_t\}$ as given. We then conduct aggregation and characterize equilibrium by a system of differential equations. Finally, we study bubbleless and bubbly equilibria.

3.1 Optimal Contract

We first solve the optimal contracting problem (3) subject to (4), (5), (6), and (7), taking r_{kt} as given. Conjecture that the stock market value of the bank takes the following form:

$$V_t(N_t) = Q_t N_t + B_t, \tag{10}$$

where Q_t and B_t are to be determined aggregate states that are independent of individual bank's characteristics. We may interpret Q_t as the shadow price of net worth. We will show below that both $B_t = 0$ and $B_t \neq 0$ can be part of an optimal contract because the contracting problem does not give a contraction mapping due to the incentive constraint (7). Because of limited liability, we only consider the solution with $B_t \geq 0$ for all t. We interpret the term B_t as a bubble component of the stock market value of the bank. We will show that both $B_t = 0$ and $B_t > 0$ can sustain in equilibrium.

We summarize the solution to the contracting problem in the following:

Proposition 1 *If $Q_t > 1$ and $r_{kt} > r$, then Q_t and B_t satisfy the following differential equations:*[2]

$$rQ_t = Q_t\left[r_{kt} + (r_{kt} - r)\,\xi Q_t\right] + \theta(1 - Q_t) + \dot{Q}_t, \tag{11}$$

[2] We use $\dot{X}_t$ to denote dX_t/dt for any variable X_t.

$$rB_t = Q_t\left(r_{kt} - r\right)B_t + \dot{B}_t, \tag{12}$$

and the transversality conditions:

$$\lim_{T\to\infty} e^{-rT} Q_T N_T = \lim_{T\to\infty} e^{-rT} B_T = 0. \tag{13}$$

The intuition behind this proposition is the following: If $Q_t > 1$, then dividend constraint (5) must bind. Paying out one dollar dividends gives the banker one dollar benefit. But retaining one dollar raises the marginal value of the bank stock by Q_t dollars. If $Q_t > 1$, then the banker prefers to reduce dividend payment until the constraint (5) binds. If $r_{kt} > r$, then the bank prefers to lend as much as possible by borrowing from depositors until the borrowing constraint (7) binds.

Given the conjectured value function in (10), we rewrite the borrowing constraint as:

$$D_t = N_t \xi Q_t + B_t. \tag{14}$$

We can then substitute the conjectured value function in (10) into the Bellman equation (3) subject to (4), (5), (6), and (14). Solving this problem and matching coefficients, we obtain differential equations (11) and (12) in which both Q_t and B_t are non-predetermined. There are two types of solutions to these differential equations. If both depositors and banks believe that the bank stock has a low value in that $B_t = 0$ for all t, then the optimal contracting problem is characterized by equation (11) only. If both depositors and banks believe that the bank stock has a high value because it contains a bubble component $B_t > 0$, then the bubble relaxes the borrowing constraint (7) and allows the bank to attract more deposits (see (14)). This allows the bank to make more loans and generates more profits and dividends, justifying the initial belief of a high value. This positive feedback loop mechanism can support a bubble.

Equation (11) is an asset-pricing equation. The left-hand side of this equation gives the return on bank capital. The right-hand side consists of dividends and capital gains. Dividends consist of returns from bank lending net of deposits payments and bank payouts.

To interpret equation (12), we rewrite it as:

$$r = \underbrace{Q_t\left(r_{kt} - r\right)}_{\text{dividend yields}} + \underbrace{\dot{B}_t / B_t}_{\text{capital gains}} \quad \text{if } B_t > 0. \tag{15}$$

This equation says that the return on the bubble is equal to the capital gains plus dividend yields. The second term on the right-hand side of the above equation represents capital gains. The first term represents dividend yields. The intuition is as follows. One dollar of the bubble allows the bank to relax the borrowing constraint by one dollar. This allows the bank to attract one more dollar of deposits and hence makes one more dollar of loans. This raises bank net by $r_{kt} - r$ dollar. Thus, the net benefit to the bank is $(r_{kt} - r)$ times the shadow price Q_t of its net worth.

The restriction on the bubble on intrinsically useless assets or on assets with exogenously given payoffs is different from (15). In particular, there is no dividend yields term. This means that the transversality condition in (13) can rule out bubbles in infinite-horizon models. By contrast, because of the dividend yields term in (15), the transversality condition cannot rule out bubbles in our model.

3.2 Equilibrium System

We now aggregate individual decision rules and impose market-clearing conditions to derive the equilibrium system.

Proposition 2 *If $Q_t > 1$ and $r_{kt} > r$, then the three variables, (B_t, Q_t, N_t), satisfy the equilibrium system, (11), (12), and*

$$\dot{N}_t = (r_{kt} - \theta + (r_{kt} - r)\,\xi Q_t)\,N_t + (r_{kt} - r)\,B_t, \tag{16}$$

where N_0 is given and

$$r_{kt} = \alpha\left((\xi Q_t + 1)\,N_t + B_t\right)^{\alpha - 1} - \delta. \tag{17}$$

The transversality condition (13) also holds.

Once we obtain B_t, Q_t and N_t, we can then derive the equilibrium capital, output, consumption, and wage as follows:

$$\begin{aligned} K_t &= (\xi Q_t + 1)\,N_t + B_t, \\ Y_t &= K_t^{\alpha},\ w_t = (1-\alpha)\,K_t^{\alpha}, \\ C_t &= K_t^{\alpha} + \delta K_t - \dot{K}_t. \end{aligned}$$

As we discussed in the previous subsection, there may be two types of equilibrium. In a bubbleless equilibrium, $B_t = 0$ for all t. In a bubbly equilibrium, $B_t > 0$ for all t. To analyze the existence of these two types of equilibrium, we first study steady state in which all aggregate variables are constant over time. We then study local dynamics around a steady state. We shall impose conditions such that $Q_t > 1$ and $r_{kt} > r$ in a neighborhood of a steady state.

3.3 First-Best Benchmark

We start with the first-best benchmark in which the lending rate is equal to the deposit rate, $r_{kt}^{FB} = r$. In addition, the shadow price of net worth is equal to one, $Q_t^{FB} = 1$. In this case, there is no borrowing constraint (7). The first-best capital stock K^{FB} is equal to a constant:

$$K^{FB} = \left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}.$$

Banks transfer deposits $D^{FB} = K^{FB}$ to non-financial firms and make zero profits. The first-best consumption is equal to

$$C^{FB} = \left(K^{FB}\right)^{\alpha} - \delta K^{FB}.$$

3.4 Bubbleless Equilibrium

Now, we introduce the borrowing constraint (7) and analyze the bubbleless equilibrium in which $B_t = 0$ for all t. We first consider steady state. We use a variable without a subscript t to denote its steady state value. Then, by Proposition 2, Q and N satisfy

$$rQ = (r_k + (r_k - r)\,\xi Q)Q + \theta(1-Q), \tag{18}$$

$$\left(r_k - \theta + (r_k - r)\,\xi Q\right) N = 0, \tag{19}$$

where

$$r_k = \alpha\left(\xi QN + N\right)^{\alpha-1} - \delta. \tag{20}$$

Solving the above equations yields:

Proposition 3 *If $\theta > r$, then a steady-state equilibrium (Q^*, N^*) without banking bubbles exists. In this equilibrium,*

$$r_k^* = r + \frac{r\,(\theta - r)}{r + \xi\theta}, \quad Q^* = \frac{\theta}{r},$$

$$N^* = \frac{1}{\xi Q^* + 1}\left(\frac{r_k^* + \delta}{\alpha}\right)^{\frac{1}{\alpha-1}}.$$

Condition $\theta > r$ ensures that $Q^* > 1$ and $r_k^* > r$ so that we can apply Proposition 2 to derive the steady state. In a steady state, equation (4) implies that

$$0 = r_k N + (r_k - r)\,D - \theta N, \tag{21}$$

where we use the fact that $C^b = \theta N$. Thus, we obtain

$$K = N + D = \frac{\theta - r}{r_k - r} N. \tag{22}$$

It follows that the steady-state leverage ratio is given by

$$\frac{K}{N} = \frac{\theta - r}{r_k - r}.$$

In a bubbleless steady state, we substitute r_k^* given in Proposition 3 into the above equation, we can show that $K/N > 1$.

Next, turn to the equilibrium dynamics which are characterized by the following system of differential equations by Proposition 2:

$$\dot{Q}_t = (r + \theta) Q_t - Q_t [r_{kt} + (r_{kt} - r) \xi Q_t] - \theta,$$

$$\dot{N}_t = (r_{kt} - \theta + (r_{kt} - r) \xi Q_t) N_t, \; N_0 \text{ given},$$

where

$$r_{kt} = \alpha ((\xi Q_t + 1) N_t)^{\alpha - 1} - \delta.$$

This nonlinear system has no closed form solution. But we can solve this system numerically using a finite-difference method. Figure 1 illustrates the phase diagram.

We can show that the steady state is a saddle point. Both isoclines $\dot{N}_t = 0$ and $\dot{Q}_t = 0$ are downward sloping. We use the phase diagram to understand why the steady state is a saddle point. We first look at the isoclines $\dot{Q}_t = 0$, which draws the combination of Q_t and N_t such that $\dot{Q}_t = 0$. Denote such combination of N_t and Q_t as $N = q(Q)$. We first show that any point on the right of the isoclines $\dot{Q}_t = 0$ satisfies $\dot{Q}_t > 0$. In fact, for any $N_t > q(Q_t)$, we have

$$\begin{aligned}
\dot{Q}_t &= (r + \theta) Q_t - Q_t [r_{kt} + (r_{kt} - r) \xi Q_t] - \theta \\
&> (r + \theta) Q_t - \theta \\
&\quad - Q_t \left[\alpha ((\xi Q_t + 1) q(Q_t))^{\alpha - 1} - \delta + \left(\alpha ((\xi Q_t + 1) q(Q_t))^{\alpha - 1} - \delta - r \right) \xi Q_t \right] \\
&= 0,
\end{aligned}$$

where the equality in the last line comes from the definition of $q(Q)$. The intuition is that for a given Q_t, a higher level of net worth will reduce the lending rate. Hence the return of the net worth from lending decreases. To achieve a required return r, there must be some additional compensation from capital gain. Conversely, any point on the left of the isoclines $\dot{Q}_t = 0$ then implies $\dot{Q}_t < 0$.

We now turn to the isocline $\dot{N}_t = 0$, which gives the combination of Q_t and N_t such that $\dot{N}_t = 0$. We denote $N = n(Q)$ for such combination. We how that $\dot{N}_t < 0$ for any point on right of the isocline $\dot{N}_t = 0$. For $N_t > n(Q_t)$, we have

$$\begin{aligned}
\dot{N}_t &= (r_{kt} - \theta + (r_{kt} - r)\,\xi Q_t)\,N_t \\
&< \left(\alpha\left((\xi Q_t + 1)\,n(Q_t)\right)^{\alpha-1} - \delta - \theta + \left(\alpha\left((\xi Q_t + 1)\,n(Q_t)\right)^{\alpha-1} - \delta - r\right)\xi Q_t\right) N_t \\
&= 0,
\end{aligned}$$

where the equality in the line comes from the definition of $n(Q)$. The intuition is as follows. For given Q, a higher level of N reduces the return on the lending rate. As a result, the net return on the net worth will be less than the dividend distribution rate θ. As a result, bank net worth must fall, i.e., $\dot{N}_t < 0$. Conversely, any point on the left of the isoclines $\dot{N}_t = 0$ implies $\dot{N}_t > 0$.

Proposition 3 shows that the two isoclines $q(Q)$ and $n(Q)$ cross at the steady state level $Q^* = \theta/r$. We now show that $q(Q) > n(Q)$ if $Q < Q^*$ and $q(Q) < n(Q)$ if $Q > Q^*$. Notice that r_{kt} can be written as a function of Q_t and N_t as $r_{kt} = r_k(Q_t, N_t)$. We can rewrite the isoclines $\dot{Q}_t = 0$ as

$$\begin{aligned}
rQ_t - \theta &= Q_t\left[r_k(Q_t, N_t) - \theta + (r_k(Q_t, N_t) - r)\,\xi Q_t\right], \\
&= Q_t \frac{\dot{N}_t}{N_t},
\end{aligned}$$

So on the isoclines $\dot{Q}_t = 0$, $\dot{N}_t / N_t > 0$ if and only if $Q_t > Q^* = \theta/r$. This implies that $q(Q) < n(Q)$ if $Q_t > Q^*$ and $q(Q) > n(Q)$ if $Q_t < Q^*$. In other words, the locus of $\dot{Q}_t = 0$ lies on the left side of $\dot{N}_t = 0$ for $Q_t > Q^*$ but on the right of $\dot{N}_t = 0$ for $Q_t < Q^*$.

Figure 1 plots the phase diagram, where the arrows indicate the direction of changes for Q_t and N_t. The saddle path is downward sloping. The intuition is the following: If the bank net worth N_t is lower than the steady state value, then the lending rate r_{kt} is relatively high. This causes the bank to raise more lending and hence the bank gradually accumulates more bank capital. This in turn lowers the lending rate r_{kt} and hence the shadow value of the net worth Q_t gradually decreases to the steady state. The case when N_t is higher than the steady state value can be similarly analyzed.

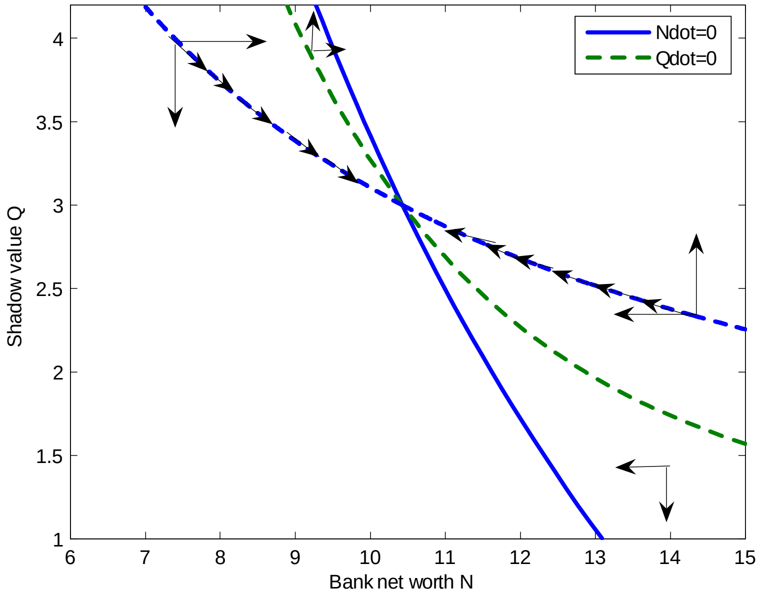

Figure 1: The phase diagram for the bubbleless equilibrium system. The $Ndot = 0$ and $Qdot = 0$ lines plot the points (N, Q) such that $\dot{N} = 0$ and $\dot{Q} = 0$ respectively. The dashed line with arrors plots the saddle path solution. We set $r = 0.01$, $\theta = 0.03$, $\delta = 0.025$, $\alpha = 0.33$, and $\xi = 0.3$.

3.5 Deterministic Banking Bubbles

Now, we tun to the equilibrium with banking bubbles in which $B_t > 0$ for all $t > 0$. By Proposition 2, the following conditions hold in the bubbly steady-state equilibrium:

$$rQ = (r_k + (r_k - r)\,\xi Q)Q + \theta(1-Q), \tag{23}$$

$$rB = Q\,(r_k - r)\,B, \tag{24}$$

$$0 = (r_k - \theta + (r_k - r)\,\xi Q)\,N + (r_k - r)\,B, \tag{25}$$

where

$$r_k = \alpha\left((\xi Q + 1)\,N + B\right)^{\alpha - 1} - \delta. \tag{26}$$

We use a variable with a superscript b to denote its bubbly steady-state value. Solving the above system of equations yields:

Proposition 4 *If*

$$0 < \xi < \frac{\theta}{r} - \frac{r}{\theta} - 1, \tag{27}$$

then both steady-state equilibria with and without banking bubbles exist. In the steady-state bubbly equilibrium,

$$Q^b = \frac{r+\theta}{\theta - \xi r} > 1, \tag{28}$$

$$r_k^b = r + \frac{r(\theta - r\xi)}{r+\theta} > r, \tag{29}$$

$$\frac{B}{N^b} = \frac{-\left(r_k^b - \theta + \left(r_k^b - r\right)\xi Q^b\right)}{r_k^b - r} > 0. \tag{30}$$

In addition, $r_k^b < r_k^*$ and hence $K^b > K^*$.

Condition (27) ensures that $\theta > r$ so that $Q^b > 1$ and $r_k^b > r$. As a result, we can apply Proposition 2 in the neighborhood of the steady state. In addition, it implies that a bubbleless equilibrium also exists by Proposition 3. Condition (27) also implies that $B/N^b > 0$.

Using Propositions 3 and 4, we can easily show that $r_k^b < r_k^*$. Thus, the capital stock in the bubbly steady-state is larger than that in the bubbleless steady state. The intuition is that banking bubbles allow banks to relax borrowing constraints and attract more deposits. Thus, banks make more loans to finance investment, leading to a raise in the capital stock.

Next, we discuss the equilibrium dynamics. There is no closed-form solution for the bubbly equilibrium system characterized in Proposition 2. The analysis of stability of the system is also complex. Thus, we present a numerical example to illustrate the solution using the same parameter values as those in Figure 1. As Proposition 4 shows, these parameter values imply that both bubbleless and bubbly equilibria exist. We find that the bubbly steady-state capital stock is equal to $K_b = 22.56$, which is larger than the bubbleless steady state capital stock $K^* = 19.81$. In addition, the bubbly state $(Q_b, B, N_b) = (1.49, 11.56, 7.61)$ is a local saddle point.[3] This implies that given an initial value (close to the steady state) for the predetermined variable N_t, there exist initial values for the non-predetermined variables Q_t and B_t such that (Q_t, B_t, N_t) converge to the steady state along a saddle path. Figure 2 plots the equilibrium solution for paths of $(N_t, Q_t, B_t, V_t,\ K_t, r_{kt}, Y_t, D_t)$ respectively. We suppose that the initial level of bank net worth is lower than the steady state value. In this case, the bank makes less lending, which makes the firm's initial capital stock lower than its steady steady level. This causes the cost of capital or the lending rate to be higher than the steady state level.

Given the parameter values in the example, Proposition 4 implies that a bubbly equilibrium exists and the bank has an incentive to create a bubble to relax the borrowing constraint. The existence of a bubble allows the bank to attract more deposits and make more lending. As a result, the capital stock rises, leading to the rise in output. As more firm capital and bank capital are gradually built up, the stock market value of the bank rises and the cost of capital and the lending rate fall over time. The banking bubble also falls over time. In the long run, all these variables converge to the steady state.

4 Stochastic Banking Bubbles

We have shown that both bubbleless and bubbly equilibria can coexist under some assumptions. We now follow Blanchard and Watson (1982) and Weil (1987) to construct an equilibrium with stochastic bubbles. Following the continuous time modeling of Miao and Wang (2011a), we assume that all agents in the economy believe that the banking bubble may burst in the future. The arrival of this event follows a Poisson process with the arrival rate π. After the bubble bursts, it cannot reappear in the future. Otherwise, if rational agents can anticipate that the bubble will reemerge in the future, there will be arbitrage opportunities following the initial burst of the bubble. To generate a new bubble, we need a new asset or a new firm to be created to carry this bubble (see Martin and Ventura (2011), Wang and Wen (2011), and Miao and Wang (2011d) for models of recurrent bubbles).

[3] The multiplicity of equilibria in our model has a different nature than the indeteminancy literature surveyed in Farmer (1999) or Benhabib and Farmer (1999).

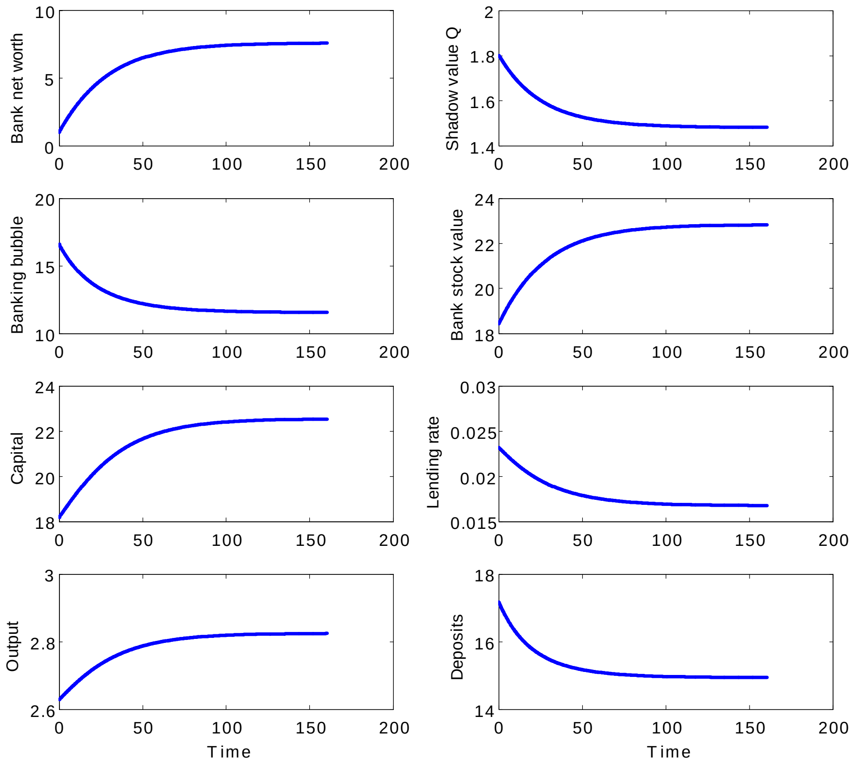

Figure 2: This figure plots the bubbly equilibrium paths of (N_t, Q_t, B_t, V_t, K_t, r_{kt}, Y_t, C_t). We set $r = 0.01$, $\theta = 0.03$, $\delta = 0.025$, $\alpha = 0.33$, and $\xi = 0.3$.

We solve for the equilibrium with stochastic bubbles by backward induction. After the burst of the banking bubble, the economy enters the bubbleless equilibrium. We have solved this equilibrium in Section 3. Here we write the bubbleless equilibrium shadow value of the net worth in a feedback form: $Q_t^* = g(N_t)$, where g is some function. We write the stock market value of the bank after the bubble bursts as $V^*(N_t, Q_t^*) = Q_t^* N_t$. Turn to the case in which the bubble has not bursted. In this case, the Bellman equation is given by

$$\begin{aligned} rV(N_t, Q_t, B_t) &= \max_{D_t,\, C_t^b} \; C_t^b + V_N(N_t, Q_t, B_t)\left(r_{kt}N_t + (r_{kt} - r) D_t - C_t^b\right) \\ &\quad + V_Q(N_t, Q_t, B_t)\dot{Q}_t + V_B(N_t, Q_t, B_t)\dot{B}_t \\ &\quad + \pi\left[V^*(N_t, Q_t^*) - V(N_t, Q_t, B_t)\right], \end{aligned} \tag{31}$$

subject to (5), (6), and (7), where (Q_t, B_t) is the aggregate state vector and N_t is the individual state. This Bellman equation actually describes an asset pricing equation for the bank stock. The left-hand side of equation (31) is the return on the bank stock. The right-hand said gives dividends C_t^b plus capital gains. Capital gains consist of four components. The first three components are due to changes in bank net worth N_t, shadow value Q_t and the bubble B_t. The last component is due to changes in beliefs. When the bubble bursts with arrival rate π, the bank value shifts from $V(N_t, Q_t, B_t)$ to $V^*(N_t, Q_t^*)$. The following proposition characterizes the solution:

Proposition 5 *If $Q_t > 1$ and $r_{kt} > r$, then the equilibrium (Q_t, B_t, N_t) before banking bubbles burst satisfies the system of differential equations:*

$$(r + \pi) Q_t = Q_t\left[r_{kt} + (r_{kt} - r)\xi Q_t\right] + (1 - Q_t)\theta + \pi Q_t^* + \dot{Q}_t, \tag{32}$$

$$(r + \pi) B_t = Q_t (r_{kt} - r) B_t + \dot{B}_t, \tag{33}$$

and (16), where r_{kt} satisfies equation (17). Here, $Q_t^* = g(N_t)$ is the equilibrium shadow price of net worth after banking bubbles burst.

As in Weil (1987) and Miao and Wang (2011a), the possibility of bubble bursting gives a risk premium so that the return on the bubble is equal to $r + \pi$ as revealed by equation (33). Since there is no closed-form solution to the system of differential equations in the proposition, we solve this system numerically using the shooting method combined with the finite difference method.

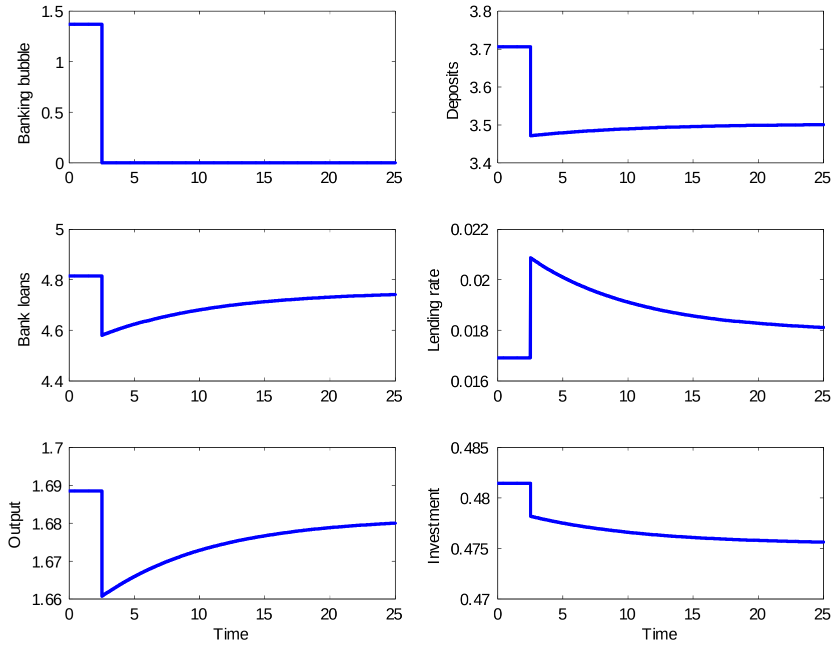

Figure 3: The transition paths when the bubble bursts at time $t = 2.5$. The parameter values are $r = 0.01$, $\theta = 0.04$, $\delta = 0.1$, $\alpha = 0.33$, and $\xi = 0.7$.

To simplify the analysis, we follow Weil (1987), Kocherlakota (2009), and Miao and Wang (2011a) and focus on a particular type of stationary equilibrium with stochastic bubbles. This equilibrium has the feature that all equilibrium variables are constant before bubbles burst. After bubbles burst, the economy then moves to the bubbleless equilibrium path. Using Proposition 5 and setting $B_t = B$, $Q_t = Q$, $N_t = N$, and $r_{kt} = r_k$ before bubbles burst, we obtain the following system of nonlinear equations:

$$(r + \pi) Q = Q [r_k + (r_k - r) \xi Q] + (1 - Q_t) \theta + \pi g(N), \tag{34}$$

$$r + \pi = Q(r_k - r), \tag{35}$$

$$0 = (r_k - \theta + (r_k - r) \xi Q) N + (r_k - r) B, \tag{36}$$

$$r_k = \alpha ((\xi Q + 1) N + B)^{\alpha - 1} - \delta. \tag{37}$$

To solve this system, we need to know the function g. This function can be obtained by solving the equilibrium after bubbles burst using the shooting and finite difference method.

Figure 3 presents a numerical example. In this example, we assume that banking bubbles burst at time $t = 2.5$. Immediately after the burst of banking bubbles, the bank's borrowing constraint tightens, causing deposits to fall discontinuously at $t = 2.5$. Deposits then gradually move to the lower bubbleless steady-state level. Due to the fall of deposits, the bank's balance sheet worsens. Thus, the bank reduces lending and hence the lending rate rises. This in turn causes non-financial firms to reduce investment and production. Thus, output falls discontinuously and then gradually moves to the lower bubbleless steady-state level. This example illustrates that even though there is no shock to fundamentals of the economy, the shift in beliefs causes the collapses of banking bubbles. This in turns causes a recession.

5 Bank Capital Requirements

Bank capital requirements determine how much liquidity is required to be held for a certain level of assets through regulatory agencies such as the Bank for International Settlements, Federal Deposit Insurance Corporation, or Federal Reserve Board. These requirements are put into place to ensure that financial institutions are not participating or holding investments that increase the risk of default and that they have enough capital to sustain operating losses while still honoring deposit withdrawals. Bank capital requirements enhance the stability of the banking system. The traditional rationale for these requirements is related to banks' risk-taking behavior. Due to limited liability and the access to secured deposits, banks have an incentive to choose risky projects which raise the probability of bank failure. Increasing the percentage of the investment funded by bank capital limits this risk taking activity for banks.

In this section, we argue that bank capital requirements may prevent the formation of banking bubbles. But if the capital requirements are too stringent, then banks will lend less and charge more for loans, thereby reducing the steady-state capital stock and efficiency. We model bank capital requirements as follows:

$$N_t \geq \lambda \left(D_t + N_t\right), \tag{38}$$

where $\lambda \in (0, 1)$ is the bank capital requirement ratio. Rewrite (38) as

$$D_t \leq \frac{1-\lambda}{\lambda} N_t. \tag{39}$$

We incorporate this inequality to study the effects of bank capital requirements on equilibrium outcomes. We suppose that condition (27) in Proposition 4 is satisfied so that both bubbleless and bubbly equilibria coexist.

Proposition 6 *Suppose that condition (27) in Proposition 4 is satisfied. Then: (i) If*

$$\frac{1-\lambda}{\lambda} > \frac{\theta - r_k^b}{r_k^b - r}, \tag{40}$$

then the bubbleless and bubbly equilibria characterized in Proposition 4 are unaffected by the bank capital requirements. (ii) If

$$0 < \frac{\theta - r_k^*}{r_k^* - r} < \frac{1-\lambda}{\lambda} < \frac{\theta - r_k^b}{r_k^b - r}, \tag{41}$$

then a banking bubble cannot exist and only the bubbleless equilibrium characterized in Proposition 3 exists. (iii) If

$$0 < \frac{1-\lambda}{\lambda} < \frac{\theta - r_k^*}{r_k^* - r}, \tag{42}$$

then a banking bubble cannot exist. The steady-state lending rate and the capital stock satisfies

$$r_k = \lambda\theta + (1-\lambda)\, r > r_k^*, \;\; K < K^*.$$

This proposition shows that if the bank capital requirement ratio is too small, then bank capital requirements are ineffective in that they do not affect the equilibrium without these requirements. Thus, they cannot prevent the existence of banking bubbles. If the bank capital requirement ratio is too large, then bank capital requirements can prevent the formation of banking bubbles. However, they lead to a high lending rate and low steady-state capital. If the bank capital requirement rate is in the intermediate range, then a banking bubble cannot exist and the economy is in the bubbleless equilibrium studied in Section 3.4.

6 Credit Policy

During the recent financial crisis, many central banks around the world, including the U.S. Federal Reserve (Fed), used their powers as a lender of last resort to facilitate credit flows. As Gertler and Kiyotaki (2010) point out, the Fed employed three general types of credit policies during the crisis. The first is discount window lending. The Fed used the discount window to lend funds to commercial banks that in turn lent them out to non-financial borrowers. The second is direct lending. The Fed lent directly in high grade credit markets, funding assets that included commercial paper, agency debt and mortgage backed securities. The third is equity injections. The Treasury coordinated with the Fed to acquire ownership positions in commercial banks by injecting equity.

For simplicity, we consider the impact of direct lending policy. We now introduce a central bank in our baseline model. Because the collapse of banking bubbles tightens banks' borrowing constraints, households reduce deposits and banks reduce lending, causing a crisis. The central bank can help credit flows by direct lending to non-financial firms. Suppose that the central bank lends $\psi_t K_t$ to non-financial firms. Suppose that ψ_t responds to the credit market condition. In particular, it responds to the credit spreads $r_{kt} - r > 0$. As we show in Section 4, during the crisis period, credit spreads rise. Thus they can be used as an indicator of economic activity. Since the riskfree rate r is constant in our model, changes in credit spreads are effectively changes in the lending rate r_{kt}. To simplify computation, assume that ψ_t takes the following feedback rule

$$\psi_t = 1 - \left(\frac{r + \delta}{r_{kt} + \delta} \right)^{\gamma} \in (0, 1), \tag{43}$$

where the parameter $\gamma > 0$ measures the strength of the feedback. This rule implies that a larger credit spread (or a larger r_{kt}) induces more direct lending. In addition, a larger value of γ implies that the central bank lends more to the non-financial firms. When $\gamma = 0$, $\psi_t = 0$ and hence the model reduces to that in Section 4 without credit policy.

Assume that direct lending is financed by lump-sum taxes. However it is costly because the central bank is less efficient at intermediating funds. Following Gertler and Kiyotaki (2010), assume that the central bank faces an efficiency cost τ per unit of lending, which may be thought of as a cost of evaluating and monitoring borrowers. For simplicity, we ignore government spending and hence the resource constraint is given by

$$C_t + I_t + \tau \psi_t K_t = Y_t.$$

Given the direct lending policy, the balance sheet equation becomes

$$K_t = N_t + D_t + \psi_t K_t.$$

This equation says that capital expenditure can be financed by private bank lending $N_t + D_t$ and central bank lending $\psi_t K_t$. It follows from the above equation and equation (43) that

$$K_t = \frac{1}{1 - \psi_t} (N_t + D_t) = \left(\frac{r_{kt} + \delta}{r + \delta} \right)^{\gamma} (N_t + D_t). \tag{44}$$

Note that above modeling of direct lending is equivalent to having the central bank channel funds to non-financial borrowers via private banks, as occurred with depository facilitates set up prior to the Lehman collapse.

We shall solve the equilibrium with stochastic bubbles and with credit policy by backward induction. We first start with the case after bubbles burst and then move back to solve the case before bubbles burst.

6.1 Equilibrium after Bubbles Burst

As in Sections 3 and 4, we can show that if $Q_t > 1$ and $r_{kt} > r$, then the constraints (5) and (7) bind. We can also show that the stock market value of the bank is given by $V_t(N_t) = Q_t N_t$. It follows that $D_t = \xi Q_t N_t$. Thus, the equilibrium system for (Q_t, K_t, N_t, r_{kt}) after bubbles burst is given by (11) and

$$\dot{N}_t = (r_{kt} - \theta + (r_{kt} - r)\,\xi Q_t)\, N_t, \tag{45}$$

$$K_t = \left(\frac{r_{kt} + \delta}{r + \delta}\right)^{\gamma} N_t\,(1 + \xi Q_t)\,, \tag{46}$$

$$r_{kt} = \alpha K_t^{\alpha - 1} - \delta. \tag{47}$$

The solution must also satisfy the standard transversality condition. Equation (45) follows from (16) by setting $B_t = 0$. Equation (46) follows from (44). The last equation (47) follows from (8) and the market-clearing condition $L_t = 1$. Using equations (46) and (47), we can explicitly solve for K_t and hence r_{kt} in terms of N_t and Q_t :

$$K_t = \left[\frac{\alpha^{\gamma} N_t\,(1 + \xi Q_t)}{(r + \delta)^{\gamma}}\right]^{\frac{1}{1+\gamma-\alpha\gamma}}. \tag{48}$$

Substituting this solution into (11) and (45), we find that the equilibrium system reduces to two differential equations for two variables (Q_t, N_t). We write the solution in a feedback form $Q_t^* = G(N_t)$ for some function G. Note that the feedback rule specified in (43) facilitates this characterization of the solution. This simplification is very useful when we analyze the case before bubbles burst in the next subsection.

We can use (11) and (45) to derive the steady-state solution:

$$Q^* = \frac{\theta}{r},\; r_k^* = r + \frac{r\,(\theta - r)}{r + \xi\theta}.$$

This solution is the same as that in the case without credit policy characterized in Proposition 3. As a result, it follows from (47) that

$$K^* = \left(\frac{1}{\alpha}\left[r + \delta + \frac{r\,(\theta - r)}{r + \xi\theta}\right]\right)^{\frac{1}{\alpha - 1}}.$$

This implies that the steady-state capital stock is also the same as that in the case without credit policy. Labor is exogenously given, it follows that credit policy also does not affect the steady-state output, consumption and the lending rate after bubbles burst.

By equation (48), we can solve for bank net worth

$$N^* = \frac{(r+\delta)^\gamma}{\alpha^\gamma} \frac{(K^*)^{1+\gamma-\alpha\gamma}}{1+\xi\theta/r}.$$

Thus, credit policy affects the steady-state net worth of the bank after bubbles burst. A larger value of γ implies that the central bank lends more to the non-financial firms, which substitute bank lending and hence lowers bank net worth. It follows that credit policy also affects the steady-state level of deposits after bubbles burst.

6.2 Equilibrium before Bubbles Burst

Turn to the case before the bubble bursts. As in Section 4, we can show that the stock market value of the bank before bubbles burst is given by

$$V_t(N_t) = Q_t N_t + B_t,$$

where $B_t > 0$ represents a bubble. If $Q_t > 1$ and $r_{kt} > r$, then the constraints (5) and (7) bind. It follows that D_t is given by (14). We can then show that the equilibrium system for $(B_t, Q_t, N_t, K_t, r_{kt})$ before bubbles burst is given by equations (32), (33), (16), (47) and

$$K_t = \left(\frac{r_{kt}+\delta}{r+\delta}\right)^\gamma \left[N_t(1+\xi Q_t) + B_t\right], \tag{49}$$

where $Q_t^* = G(N_t)$ in equation (32). The first four equations are the same as those derived before. The last equation (49) follows from the new balance sheet equation (44) and the binding borrowing constraint.

As in Section 4, we study a particular type of stationary equilibrium with stochastic bubbles and credit policy. This equilibrium has the feature that all equilibrium variables are constant before bubbles burst. After bubbles burst, the economy then moves to the bubbleless equilibrium path. Using equations (32), (33), (16), (47), and (49), we can show that the equilibrium solution for (B, Q, N, K, r_k) before bubbles burst satisfy the following system of nonlinear equations: (35), (36), and

$$(r+\pi) Q = Q\left[r_k + (r_k - r)\xi Q\right] + (1-Q_t)\theta + \pi G(N), \tag{50}$$

$$K = \left\{\frac{\alpha^\gamma}{(r+\delta)^\gamma}\left[N(1+\xi Q) + B\right]\right\}^{\frac{1}{1+\gamma-\alpha\gamma}}, \tag{51}$$

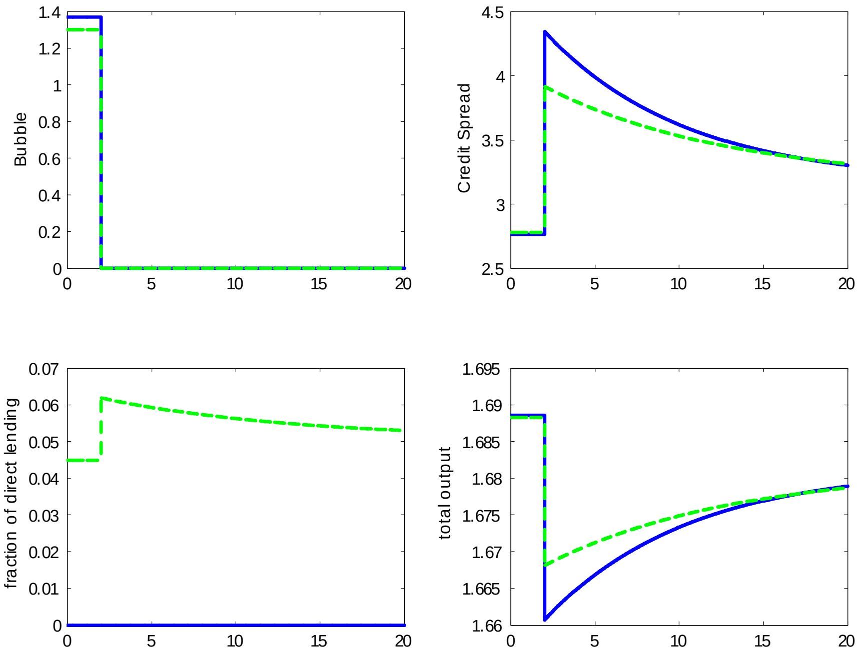

Figure 4: Transition paths when the bubble bursts at time $t = 2.5$. Dashed lines describe the case without credit policy. Solid lines describe the case with credit policy. For this case, we set $\gamma = 0.75$ and $\tau = 0.01$. Credit spreads are measured in basis point. The parameter values are $r = 0.01$, $\theta = 0.04$, $\delta = 0.1$, $\alpha = 0.33$, and $\xi = 0.7$.

$$r_k = \alpha K^{\alpha-1} - \delta.$$

Notice that this system is similar to that described in Section 4. There are two differences. First, $G(N)$ appears in (50), while $g(N)$ appears in (35). This is because credit policy changes the bubbleless equilibrium path for Q_t. The second difference is that credit policy gives a new balance sheet equation (51). To solve the above system, one needs to know the function $G(N)$. As in Section 4, we can solve for it using the the shooting and finite difference method.

Figure 4 presents a numerical example. We assume that banking bubbles burst at time $t = 2.5$. Dashed lines describe the equilibrium paths with credit policy, while solid lines describe the equilibrium paths without credit policy. This figure reveals that credit policy can reduce the size of banking bubbles. Immediately after the collapse of banking bubbles, credit spreads rise. The feedback rule of credit policy implies that the central bank should increase direct lending to the non-financial firms. As credit spreads gradually return to the bubbleless steady-state level, the fraction of direct lending ψ_t gradually decreases to the steady-state level. The central bank's direct lending substitutes part of the commercial bank's lending and hence deposits in the case with credit policy are smaller than those without credit policy. However, total lending K_t to the non-financial firms is higher in the case with credit policy. This causes the fall of output in the case with credit policy to be less severe than that in the case without credit policy. In terms of welfare as measured by consumption units, there are two effects. First, credit policy mitigates the fall of output and investment after the collapse of banking bubbles, which benefits households. Second, credit policy induces an efficiency loss in terms of output. The net effect depends on which one dominates.

7 Conclusion

We have developed a continuous-time macroeconomic model with a banking sector in which banks face endogenous borrowing constraints. There is no uncertainty about economic fundamentals. A positive feedback loop mechanism generates a banking bubble. Changes in people's beliefs can cause the collapse of the banking bubble. Immediately after the collapse of the banking bubble, households withdraw deposits and the bank reduces lending to non-financial firms. Consequently, non-financial firms cut back investment, causing output to fall. Credit policy that responds to credit spreads in a feedback rule can mitigate economic downturns. But it incurs an efficiency cost. Bank capital requirements can prevent the formation of a banking bubble. However, they limit leverage and hence reduce lending, causing investment and output to fall.

Our model is stylized and can be generalized in many dimensions. First, it would be interesting to introduce uncertainty in the model. It is likely that both uncertainty and beliefs are important during financial crises. Second, it is straightforward to introduce risk aversion and endogenous labor choice in the model. This makes the model more realistic, but complicates the analysis without changing our key insights. Third, it would be interesting to introduce recurrent banking bubbles (e.g., Martin and Ventura (2011), Wang and Wen (2011), and Miao and Wang (2011d)) and study the quantitative implications of banking bubbles.

Appendix

A Proofs

Proof of Proposition 1: We write the bank's value function as $V_t(N_t) = V(N_t, Q_t, B_t)$. By standard dynamic programming theory, it satisfied the Bellman equation:

$$\begin{aligned} rV(N_t, Q_t, B_t) &= \max_{C_t^b, D_t} \; C_t^b + V_N(N_t, Q_t, B_t)\left(r_{kt}N_t + (r_{kt} - r)D_t - C_t^b\right) \\ &\quad + V_Q(N_t, Q_t, B_t)\dot{Q}_t + V_B(N_t, Q_t, B_t)\dot{B}_t, \end{aligned}$$

subject to constraints (5), (6) and (7). We conjecture that the value function takes the form in (10). Substituting this conjecture into the above Bellman equation yields:

$$r(Q_tN_t + B_t) = \max_{C_t^b, D_t} \; C_t^b + Q_t\left(r_{kt}N_t + (r_{kt} - r)D_t - C_t^b\right) + N_t\dot{Q}_t + \dot{B}_t.$$

By this equation, if $Q_t > 1$, then constraint (5) binds so that $C_t^b = \theta N_t$. If $r_{kt} > r$, then the borrowing constraint (7) binds so that

$$D_t = V_t(N_t) = N_t\xi Q_t + B_t.$$

Substituting the solution for C_t^b and D_t into the above Bellman equation yields:

$$\begin{aligned} r(Q_tN_t + B_t) &= Q_t\left[r_{kt}N_t + (r_{kt} - r)(N_t\xi Q_t + B_t)\right] + (1 - Q_t)\theta N_t \\ &\quad + N_t\dot{Q}_t + \dot{B}_t. \end{aligned} \tag{A.1}$$

Matching coefficients for N_t and the remaining terms on the two sides of the equation yields (11) and (12). Q.E.D.

Proof of Proposition 2: By Proposition 1, if $Q_t > 1$ and $r_{kt} > r$, then Q_t and B_t satisfy equations (11) and (12). Substituting the solution $C_t^b = \theta N_t$ and $D_t = N_t\xi Q_t + B_t$ into the flow of funds constraint (4) yields (16). Equation (17) follows from (8), (2), and the market clearing condition $L_t = 1$. Q.E.D.

Proof of Proposition 3: It follows from (19) that $(r_k - r)\xi Q = \theta - r_k$. Substituting this expression into equation (18) yields $Q^* = \theta/r$. We the use equations (19) and (20) to solve for r_k^* and N^*, respectively. Q.E.D.

Proof of Proposition 4: By equation (24),

$$r = (r_k - r) Q. \tag{A.2}$$

Substituting this equation into (23) yields:

$$rQ = r_k Q + \xi r Q + \theta(1 - Q). \tag{A.3}$$

Combining the above two equations yields (28). Condition (27) ensures $\theta > \xi r$ so that $Q^b > 1$. Substituting (28) into (A.2) yields (29). Again condition (27) ensures that $r_k > r$. Substituting equations (28) and (29) into (25), we obtain equation (30). We can check that $B > 0$ if and only if condition (27) holds.

Using equation

$$\alpha K^{\alpha-1} = r_k^b + \delta,$$

we can solve for the bubbly steady-state aggregate capital stock K^b. Note that equation (22) also holds for the bubbly equilibrium. Substituting r_k^b in (29) and K^b into this equation, we can solve for the bubbly steady-state net worth N^b. Finally, we use equation (30) to solve for the steady-state bubble B.

Proof of Proposition 5: Conjecture that the value function before bubbles burst takes the following form:

$$V(N_t; Q_t, B_t) = Q_t N_t + B_t.$$

Thus, the borrowing constraint becomes

$$D_t \leq \xi Q_t N_t + B_t.$$

Substituting the above conjecture into the Bellman equation (31) yields:

$$\begin{aligned} r(Q_t N_t + B_t) &= \max_{C_t^b, D_t} \; C_t^b + Q_t \left(r_{kt} N_t + (r_{kt} - r) D_t - C_t^b \right) \\ &\quad + \pi \left[Q_t^* N_t - (Q_t N_t + B_t) \right] + N_t \dot{Q}_t + \dot{B}_t \end{aligned}$$

By this equation, if $Q_t > 1$, then the constraint (5) binds so that $C_t^b = \theta N_t$. If $r_{kt} > r$, then the borrowing constraint binds so that

$$D_t = \xi Q_t N_t + B_t.$$

Substituting these binding constraints into the Bellman equation yields:

$$\begin{aligned} r\left(Q_t N_t + B_t\right) &= \left(1 - Q_t\right)\theta N_t + Q_t r_{kt} N_t + Q_t\left(r_{kt} - r\right)\left(\xi Q_t N_t + B_t\right) \\ &\quad + \pi\left[Q_t^* N_t - \left(Q_t N_t + B_t\right)\right] + N_t \dot{Q}_t + \dot{B}_t. \end{aligned}$$

Matching coefficients of N_t and terms not related to N_t on the two sides of this equation delivers the equations given in Proposition 5. Q.E.D.

Proof of Proposition 6: Ignore capital requirements given in (39) for now. Then by assumption, the bubbly and bubbleless equilibria coexist. The borrowing constraint for the bubbly equilibrium is given by

$$D_t \leq \xi Q_t N_t + B_t.$$

In the bubbly steady state,

$$\frac{D^b}{N^b} \leq \xi Q^b + \frac{B}{N^b} = \frac{\theta - r_k^b}{r_k^b - r}.$$

where the equality follows from Proposition 4. In the bubbleless steady state, Proposition 3 implies that

$$\frac{D^*}{N^*} \leq \xi Q^* = \frac{\theta - r_k^*}{r_k^* - r}.$$

By Proposition 4,

$$\frac{\theta - r_k^b}{r_k^b - r} > \frac{\theta - r_k^*}{r_k^* - r}.$$

Now we introduce capital requirements given in (39). If condition (40) is satisfied, then the capital requirements are ineffective in the neighborhood of the bubbly steady state. If condition (41) holds, then the capital requirement constraint (39) binds in the neighborhood of the bubbly steady state, which prevents the existence of the bubbly equilibrium. But it does not bind around the bubbleless steady state characterized in Proposition 3. Thus, a bubbleless equilibrium around that steady state still exists. Finally if condition (42) holds, then the capital requirement constraint (39) binds. The new steady state is given by

$$\frac{D}{N} = \frac{1 - \lambda}{\lambda}. \tag{A.4}$$

In the steady state, equation (4) becomes

$$r_k N + \left(r_k - r\right) D - C^b = 0.$$

Since $C^b = \theta N$,

$$\frac{D}{N} = \frac{\theta - r_k}{r_k - r}. \tag{A.5}$$

Using (A.4) and (A.5) yields

$$r_k = \lambda\theta + (1 - \lambda)\, r.$$

Condition (42) implies that $r_k > r^*$ and hence $K < K^*$. Q.E.D.

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