Smooth Trading with Overconfidence and Market Power

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Abstract

This paper presents a continuous time model of oligopolistic trading among symmetric traders who agree to disagree concerning the precision of continuous flows of private information. Although traders do not share a common prior, they apply Bayes law consistently. If there is enough disagreement among traders, an equilibrium exists in which prices reveal the average of all traders’ signals immediately, but prices do not follow a martingale and traders trade on their information slowly. Each trader believes that the price is a linear function of his inventory, the derivative of his inventory, and an average of other traders’ private information. The speed with which traders adjust inventories results from a trade-off between incentives to slow down trading to reduce market impact costs in an imperfectly resilient market and incentives to speed up trading to profit from perishable information with limited half-life. Trading modest quantities much faster than consistent with equilibrium strategies results in price spikes followed by reversals.

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In real world trading, the market impact costs of dumping large quantities on the market suddenly can be much greater than the impact costs of trading a similar quantity gradually over a longer period of time. When trading based on private information decaying over time, the optimal execution of trades involves a tradeoff between slowing down the execution of trades to reduce temporary impact costs and speeding up the execution of trades to better exploit perishable private information.

The purpose of this paper is to present a dynamic model of informed trading that derives endogenously the speed with which traders trade. To model market impact, each trader is assumed to optimize his trading, taking into account his effect on prices. To model information decay, each trader is assumed to have a continuous flow of new private information about the unobserved mean-reverting growth rate of cash flows; one trader’s information decays as other traders acquire and trade on similar information. To motivate trade, we assume that traders do not share a common prior, but agree to disagree about the informativeness of their signals; each trader believes his own information is more precise than other traders believe it to be. To keep matters simple, we assume informed oligopolistic traders with the same degree of risk aversion disagree in a symmetric manner. The assumptions of exponential utility and linear Gaussian information processes about future rates of dividend growth lead to an equilibrium with a linear structure. The model is set in continuous time to make transparent the idea that each trader trades “smoothly” in the sense that the inventory of each trader is a differentiable function of time.

Unlike Grossman and Stiglitz (1980) or Kyle (1985), there are no noise traders and market makers. In the special case where traders believe other traders’ signals are completely uninformative, the model implements the idea of Black (1986) of “trading on noise as if it were information.” In the more general case where traders believe other traders’ signals have some information, each trader believes that other traders “overtrade” on the basis of their private information, as in Kyle and Lin (2011) and Scheinkman and Xiong (2003).

In modeling trading, economists face modeling trade-offs in deciding whether to use one-period models, multi-period models with a finite number of time periods, infinite-horizon discrete time models with an infinite number of time periods, continuous-time
models with a finite horizon, or continuous-time models with an infinite time horizon. For the points being made in this paper, continuous-time models make the exposition far simpler and far more intuitive than discrete-time models. Models with an infinite-time horizon result in a steady-state equilibrium that leads to more meaningful concepts of depth and price volatility. We present therefore a dynamic continuous-time infinite-horizon model with overconfidence and market power.

We look for a steady state symmetric linear equilibrium in which each trader applies Bayes law correctly given his beliefs and the dynamic equilibrium trading strategies of other traders. Each trader correctly takes into account his market impact, including how his trading affects the beliefs and trading of other traders. A symmetric linear equilibrium can be characterized based on a solution of six quadratic polynomial equations in six unknowns, which we solve numerically. There exists an obvious no-trade equilibrium with an undefined price: If each trader believes that all other traders will trade a zero quantity, it is not optimal for them not to trade.

To obtain an equilibrium with positive trading volume, there needs to be “enough” disagreement. The intuition for this condition can be understood from a one-period version of the model. The one-period model incorporates “bid-shading” in a manner similar to Rostek and Weretka (2012). Traders exploit their market power by trading approximately one-half of the amount they believe would fully reveal their information. Other traders are willing to take the opposite side of these trades only if they believe this information is “more than fully” incorporated into prices. An equilibrium with linear strategies exists only if traders believe that their signals are approximately more than twice as precise as other traders believe them to be. Our numerical solution suggests that similar condition on the degree of overconfidence is necessary in the continuous-time model. In that condition, a multiple of two appears because each trader tries to “walk” the residual demand schedule as a perfectly discriminating monopolist and the average trade price reflects only about half the price impact of the entire quantity traded.

A symmetric equilibrium of the continuous-time model has a simple and intuitive form. When all traders slow down the rate at which they trade to reduce market impact costs, this has a profound effect on the way in which liquidity is supplied.
To understand this effect intuitively, consider the continuous-time version of the model of Kyle (1985). The order flow consists of the “smooth” order flow from informed trader and the “diffusion” order flow from noise traders. Market makers in aggregate offer a static, linear, upward-sloping supply curve with the constant market depth. This implies that the price is a function of the level of inventories market makers hold. The market provides continuous liquidity for orders of all sizes. Price fluctuations are small when traders execute a small number of shares and large when they execute larger quantities.

Liquidity in the model described in this paper is very different from Kyle (1985). As in Kyle (1989), traders submit demand schedules, but net demand schedules for flows (derivatives of inventories) rather than stocks of the asset. Each trader trades in the direction of his signal and provides liquidity to other traders by trading against their information. He calculates his target inventories based on the risk aversion and the difference between his own valuation and the average valuation of others. Each trader correctly believes that the price level is a linear function of the level and the derivative of his inventory. Since trading a nontrivial quantity over a very short period of time results in very high market impact costs, each trader adjusts his inventory towards its target level only gradually. The rate of partial adjustment is derived endogenously. The speed of execution of trades is determined by tradeoffs between the half-life of private information and price resiliency. Since inventory levels are differentiable functions of time, the model has a meaningful concept of trading volume. In contrast in Kyle (1985), the costs of trading a given amount does not depend on the speed of trading when the informed trader changes his inventory level continuously, and trading volume is infinite since the inventories of noise traders follow the Brownian motion.

Concepts of depth, tightness, and resiliency from Black (1971) play out differently in our model than in Kyle (1985). Our market with smooth trading has no instantaneous depth, tight spreads if traders are willing to trade slowly, and “liquidity” which depends mostly on resiliency of security prices. Resiliency depends on the rate at which traders update their beliefs based on new public and private information. More mean-reversion in fundamentals and greater precision of private information make markets more resilient.

Our model implies that too fast trading may destabilize prices and result in sharp price
changes followed by price reversals, as aggressive trading slows down and no information arrives to support prices at the new level. Similar patterns are often observed in financial markets as, for example, the price dynamics in response to George Soros’ trades in October 1987 and the flash crash in May 2010, described in Kyle and Obizhaeva (2013). In both cases, prices plummeted rapidly as traders started to dump large quantities into the market and then rapidly recovered after the heavy selling slowed down.

Our model implements in a precise mathematical manner ideas about market liquidity describe informally by Black (1995), who envisioned a future frictionless market for exchanges as “an equilibrium in which traders use indexed limit orders at different levels of urgency but do not use market orders or conventional limit orders.” In that equilibrium, there will be no conventional liquidity available for market orders and conventional limit orders. Placement of indexed orders onto the market will move the price by an amount increasing in level of urgency. Our model effectively verify this intuition of Fisher Black.

Modern financial markets seem to become looking more and more similar to “ideal” exchanges, described by Fischer Black. Recent developments such as reduction in tick size, introduction of electronic interfaces, and emergence of algorithmic trading have facilitated “order shredding,” i.e., breaking large trades into many small pieces which are sent into the market sequentially. For example, Kyle, Obizhaeva and Tuzun (2012) find that a large fraction of reported trading volume in year 2008 was executed in 100-share trades. The wide use of order shredding makes trading strategies resemble a discrete approximation to optimal smooth trading strategies in our model.

Our model explains the apparent short-term nature of trading, even though the private information which motivates trading may have a long-term focus. If a trader acquires bullish private information which other traders do not have, he develops a more bullish estimate of the value of the asset and buys from the other traders. As other traders learn the same thing from innovations in their private information, these positions tend to be unwound. Even though the underlying information is about long term cash flow growth rates, trading positions based on such information can have a very short half life if private signals have high precision.

In our symmetric model, the equilibrium price is a linear function of the publicly
observed current dividend and the average estimate of the unobserved mean-reverting dividend growth rate across all traders. Prices are fully revealing, i.e., the current dividend level and the current price fully reveal to each trader a sufficient statistic for all they care to know about other traders’ private information. Although prices adjust to reveal new information immediately, quantities do not adjust so quickly. Even after information is already incorporated into prices, traders continue to trade on its basis because they disagree about its implications for future cash flows.

Comparing to the simple Gordon’s formula, the coefficient on the publicly observed dividend is the same, but the coefficient on the average estimate is smaller. The intuition of why prices are less sensitive to the information flow than in a similar model with no agreement to disagree is related to the intuition of the ‘beauty contest,” described by Keynes (1936). Even though traders are rational investors with a long-term horizon, they also take into account expectations of short-term price dynamics. Everybody knows that others are wrong and will soon revise their estimates, so everybody trades against others, as a result dampening the overall effect of information on prices. Making “beauty contest” endogenous leads to less volatile prices, in contrast to the famous conclusion of Keynes (1936) about excessive price volatility. Our result is consistent with Allen, Morris and Shin (2006), who also find that prices in a beauty contest react sluggishly to changes in fundamentals due to a very similar intuition: The average of martingales is not a martingale.

There is no representative agent, i.e., there are no symmetric beliefs about precision of signals such that the equilibrium price correctly reflects information. It is impossible to assign symmetric precisions to all signals to match simultaneously both the current level of the average estimate of a dividend growth rate and its dynamics, because the average of martingales is not a martingale. Regardless of beliefs, the equilibrium price does not follow a martingale. Everyone would find market anomalies such as momentum or price reversals. This issue does not exist in a one-period model, but arises in dynamic setting.

The result that traders continue to trade after price reveals their information is in sharp contrast to the intuition of Milgrom and Stokey (1982), who suggest that traders will not want to keep trading as soon as their inventories properly reflect disagreement
and prices are fully revealing.

Our model is designed to capture the conventional Wall Street wisdom that speed of trading affects prices. The empirical studies such as Chan and Lakonishok (1995), Keim and Madhavan (1997), Dufour and Engle (2000) uniformly support this intuition. Furthermore, Holthausen, Leftwich and Mayers (1990) have measured temporary and permanent price effects associated with block trades and found most of the adjustment occurring during the very first trade in a sequence, somewhat consistent with instantaneous price adjustment in our model. Almgren et al. (2005) have calibrated price impact functions depending on the speed of trading, with the functional form similar to the one endogenously derived in our model. Kyle and Obizhaeva (2013) suggest that price impact functions for various assets may be described just by a few parameters, if invariance principles are imposed.


Our model is most close to Vayanos (1999), a discrete-time dynamic model with strategic traders who get endowment shocks and trade to share the risk. The endowment shocks are themselves a form of private information. One reason traders smooth their trading is to hide the size of the endowment shock from other traders, so they do not front-run it. By trading on the endowment shock slowly traders optimize their impatience based on risk aversion—as opposed to decay of private information as in our model—against their desire to keep their private information secret. As in our paper, the conventional liquidity disappears in the model of Vayanos (1999), as time periods converge to zero. In
contrast, the model of Vayanos (1999) converges to the competitive case when the number of traders goes to infinity, since the total risk bearing capacity in the economy becomes infinite; our model does not converge to the competitive equilibrium, since traders continue to disagree even in the limit. Additionally, the market prices follow a martingale in the model of Vayanos (1999), whereas price anomalies endogenously arise due to complicated dynamics of information structure in our model market.

The remainder of this paper is structured as follows. Section I presents a one-period model of trading with overconfidence and market power. Section II outlines a fully fledged dynamic continuous-time model. Section III examines properties of prices and trades in smooth trading equilibrium. Section IV explores implications for dynamic properties of liquidity. Section V concludes. All the proofs are in the Appendix.

I. One-period Model

To develop the intuition of how equilibrium prices and quantities depend on the interaction between overconfidence and market power, we start with a one-period model. There are $N$ traders who trade a risky asset with liquidation value $\tilde{v} \sim N(0, 1/\tau_n)$ against a safe numeraire asset with liquidation value of one in order to maximize their expected constant absolute risk aversion (CARA) utility from the terminal wealth on date 1. All traders have a risk version $A$. Each trader $n$ has initial inventory of $S_n$ shares of a risky asset. Since a risky asset is in zero net supply, the sum of all inventories is zero.

Bayesian Updating. All traders observe a public signal $\tilde{i}_0 = \tilde{v} + \tilde{e}_0$ with $\tilde{e}_0 \sim N(0, 1/\tau_0)$. There are $N$ private signals $\tilde{i}_n = \tilde{v} + \tilde{e}_n$ with $\tilde{e}_n \sim N(0, 1/\tau_n)$ and $n = 1, \ldots, N$. The stock payoff $\tilde{v}$, the public signal error $\tilde{e}_0$, and $N$ private signal errors $\tilde{e}_1, \ldots, \tilde{e}_N$ are independently distributed. Trader $n$ observes signal $n$ privately, but the equilibrium discussed below fully reveals the average of other traders’ signals defined by $\tilde{i}_{-n} := \frac{1}{N-1} \sum_{m \neq n} \tilde{i}_m$. Traders agree about the precision of the public signal $\tau_0$ but disagree about the precisions of private signals. Traders are “relatively overconfident” in that trader $n$ believes $\tau_n = \tau_H$ and $\tau_m = \tau_L, m \neq n$, with $\tau_H > \tau_L \geq 0$. All traders agree to disagree about precision of
their signals.

Let $E_n$ and $Var_n$ denote trader $n$’s expectation and variance operators conditional on observing all signals $i_0, i_1, \ldots, i_N$. Using formulas for conditional expectation and variance of normal random variables, we define

$$
\bar{\tau} := Var_n^{-1}\{\tilde{v}\} = \tau_v + \tau_0 + \tau_H + (N - 1)\tau_L,
$$

then

$$
E_n\{\tilde{v}\} = \frac{\tau_0}{\bar{\tau}} \cdot \tilde{i}_0 + \frac{\tau_H}{\bar{\tau}} \cdot \tilde{i}_n + \frac{(N - 1)\tau_L}{\bar{\tau}} \cdot \tilde{i}_{-n}.
$$

Suppose there is an economist who embarks on studying properties of securities prices. In a symmetric model, an economist will probably assign the same precision to all private signals. If an economist thinks that public signal has precision $\tau_0$ and all private signals have a precision $\tau_e$ with the total precision $\tau^E = \tau_v + \tau_0 + N \cdot \tau_e$, then the conditional expectation and variance of $\tilde{v}$ given his beliefs can be easily calculated using the equations above.

There are two assumptions that can be made to model overconfidence. In “relative overconfidence” case, even though oligopolistic traders think they observe a more precise signal, they agree with each other and with an economist on what the total precision in information flow is, $\bar{\tau} = \tau^E$, i.e., $\tau_e = (\tau_H + (N - 1)\tau_L)/N$. The relative overconfidence is the belief that “I am smarter than others think I am.” In “absolute overconfidence” case, traders think there is more information in the market than an economist thinks, $\bar{\tau} > \tau^E$, i.e., $\tau_e < (\tau_H + (N - 1)\tau_L)/N$. The absolute overconfidence is the belief that “there is more information in the market than the economist thinks.” These concepts of overconfidence should not be confused with the concept of “over-optimism,” when each trader agrees with others about the precision of his signal, but thinks it has a higher mean (outside of the current model).

Utility Maximization with Market Power. Traders are imperfect competitors who explicitly take into account the effect of their trading on prices. Suppose trader $n$ believes the price is a function of the quantity he trades, $p = P(x_n)$. As a result of trading $x_n$,
trader $n$ thinks that his terminal wealth $\tilde{W}_n = \tilde{v} \cdot (S_n + x_n) - P(x_n) \cdot x_n$ will be distributed as a normal random variable with the mean and variance defined as,

$$E_n\{\tilde{W}_n\} = E_n\{\tilde{v}\} \cdot (S_n + x_n) - P(x_n) \cdot x_n,$$

(3)

$$Var_n\{\tilde{W}_n\} = (S_n + x_n)^2 \cdot Var_n\{\tilde{v}\}.$$  

(4)

Each trader $n$ then maximizes the exponential utility of his wealth,

$$E_n\{-e^{-A \cdot \tilde{W}_n}\} = -\exp\left( - A \cdot E_n\{\tilde{W}_n\} + \frac{1}{2} A^2 \cdot Var_n\{\tilde{W}_n\} \right).$$  

(5)

Plugging equations (1), (2), (3) and (4) into equation (5) yields that optimization problem is equivalent to maximizing monotonically transformed expected utility $-\frac{1}{A} \ln\left( - E\{-e^{-A \cdot \tilde{W}_n}\} \right)$ by choosing the quantity $x_n$ to trade,

$$\max_{x_n} \left( \frac{\tau_0}{\tau} \cdot \tilde{i}_0 + \frac{\tau_H}{\tau} \cdot \tilde{i}_n + \frac{(N-1)\tau_L}{\tau} \cdot \tilde{i}_{-n} \right) \cdot (S_n + x_n) - P(x_n) \cdot x_n - \frac{1}{2\tau} A \cdot (S_n + x_n)^2.$$  

(6)

Note that for a perfect competitor, where $P(x_n)$ is just a constant $p$, the optimal demand of trader $n$ would be $x_n = \frac{\tau}{A} (\frac{\tau_0}{\tau} \cdot \tilde{i}_0 + \frac{\tau_H}{\tau} \cdot \tilde{i}_n + \frac{(N-1)\tau_L}{\tau} \cdot \tilde{i}_{-n} - p) - S_n$. In contrast, an imperfect competitor recognizes that his trading may change the equilibrium price.

**Linear Conjectured Strategies.** As in Kyle (1989), we assume a single-price auction in which traders submit demand schedules $X_n(i_0, i_n, S_n, p)$ to an auctioneer, who then calculates a market clearing price $p$. Suppose trader $n$ conjectures that the other $N-1$ traders submit symmetric linear demand schedules

$$X_m(i_0, i_m, S_m, p) = \alpha \cdot i_0 + \beta \cdot i_m - \gamma \cdot p - \delta \cdot S_m, \quad m \neq n.$$  

(7)

From the market clearing condition $\sum_{m=1}^N X_m(i_0, i_m, p) = 0$ and the linear specification of demand for traders $m \neq n$, it follows that $x_n + \sum_{m \neq n} (\alpha \cdot i_0 + \beta \cdot i_m - \gamma \cdot p - \delta \cdot S_m) = 0$. Since
\[ \sum_{m=1}^{N} S_m = 0, \]
solving for \( p \) as a function of \( i_0, i_{-n}, S_n, x_n \) yields price impact function

\[ P(i_0, i_{-n}, S_n, x_n) = \frac{\alpha}{\gamma} \cdot i_0 + \frac{\beta}{\gamma} \cdot i_{-n} + \frac{1}{(N-1)\gamma} \cdot x_n + \frac{\delta}{(N-1)\gamma} \cdot S_n. \tag{8} \]

Plugging equation (8) into equation (6), we use the first order condition to find trader \( n \)'s optimal demand,

\[ x_n = \left( \frac{\tau_0}{\tau} \cdot i_0 + \frac{(N-1)\tau_L}{\tau} \cdot i_{-n} \right) - \left( \frac{\alpha}{\gamma} \cdot i_0 + \frac{\beta}{\gamma} \cdot i_{-n} \right) - \left( \frac{\delta}{(N-1)\gamma} + \frac{\lambda}{\tau} \right) \cdot S_n, \tag{9} \]

under the assumption that trader \( n \) knows the value of \( \hat{i}_{-n} \). In this equation, the first term in the numerator is trader \( n \)'s expectation of the liquidation value, the second term in the numerator is the market clearing price when trader \( n \) trades a quantity of zero and has no inventory, the last term in the numerator is the adjustment for exiting inventory, the first term and the second term in the denominator captures how trader \( n \) restricts the quantity traded due to market power and risk aversion, respectively.

If trader \( n \) does not observe \( \hat{i}_{-n} \), he may nevertheless be able to implement this optimal strategy inferring it from the market clearing price. Define \( D_X := \frac{1}{(N-1)\gamma} + \frac{\lambda}{\tau} + \frac{\lambda}{\tau} \cdot \alpha \beta \).

Solving for \( i_{-n} \) instead of \( p \) the market clearing condition with linear conjectured strategies for the other traders, substituting this solution into (9), and then solving for \( x_n \), we derive a demand schedule \( X_n(i_0, i_{-n}, S_n, p) \) for trader \( n \) as a function of price \( p \),

\[ X_n(i_0, i_{-n}, S_n, p) = \frac{1}{D_X} \cdot \left( \left( \frac{\tau_0}{\tau} - \frac{(N-1)\tau_L}{\tau} \cdot \frac{\alpha}{\beta} \right) \cdot i_0 + \frac{\tau_H}{\tau} \cdot i_{-n} + \left( \frac{(N-1)\tau_L}{\tau} \cdot \frac{\gamma}{\beta} - 1 \right) \cdot p - \left( \frac{\tau_L}{\tau} \cdot \frac{\delta}{\beta} + \frac{\lambda}{\tau} \right) \cdot S_n \right). \tag{10} \]

In a symmetric linear equilibrium, the strategy chosen by trader \( n \) is the same as the linear strategy (7) conjectured for the other traders. Equating coefficients of variables \( i_0, i_{-n}, P \) and \( S_n \) yields the system of four equations,

\[ \alpha = \frac{1}{D_X} \left( \frac{\tau_0}{\tau} - \frac{(N-1)\tau_L}{\tau} \cdot \frac{\alpha}{\beta} \right), \quad \beta = \frac{1}{D_X} \frac{\tau_H}{\tau}, \tag{11} \]
\[ \gamma = -\frac{1}{DX} \left( \frac{(N - 1)\tau_L \gamma}{\bar{\tau}} - 1 \right), \quad \delta = \frac{1}{DX} \left( \frac{\tau_L \delta}{\bar{\tau}} + \frac{A}{\bar{\tau}} \right). \]  

(12)

Its unique solution in terms of four unknowns \( \alpha, \beta, \gamma, \delta \) is

\[ \beta = \frac{(N - 2)\tau_H - 2(N - 1)\tau_L}{A(N - 1)}, \]  

(13)

\[ \alpha = \frac{\tau_0}{\tau_H + (N - 1)\tau_L} \cdot \beta, \quad \gamma = \frac{\bar{\tau}}{\tau_H + (N - 1)\tau_L} \cdot \beta, \quad \delta = \frac{A}{\tau_H - \tau_L} \cdot \beta. \]  

(14)

**Theorem 1** If \((N - 2)\tau_H - 2(N - 1)\tau_L > 0\), then there is a symmetric linear equilibrium.

1. **Trader n’s optimal equilibrium demand is**

\[ x_n^* = D_H \cdot (\tilde{i}_n - \tilde{i}_{-n}) - \delta \cdot S_n, \]  

(15)

2. **The equilibrium price is**

\[ P^* = \frac{\tau_0}{\bar{\tau}} \cdot \tilde{i}_0 + \frac{\tau_H + (N - 1)\tau_L}{N\bar{\tau}} \cdot (\Sigma_{m=1}^N \tilde{i}_m). \]  

(16)

where \( D_H := ((N - 2)\tau_H - 2(N - 1)\tau_L)/(AN) \).

The second order condition is equivalent to the denominator of equation (9) being positive, i.e., \( \frac{2}{(N - 1)\gamma} + \frac{\delta}{\bar{\tau}} > 0 \). Plugging in the solution for \( \gamma \) implies that the second order condition holds if and only if \((N - 2)\tau_H - 2(N - 1)\tau_L > 0\). A symmetric linear equilibrium does not exist unless \( N \geq 3 \) and \( \tau_H \) is sufficiently more than twice as large as \( \tau_L \). The same condition ensures that the price impact of trading is positive, i.e., \( \gamma > 0 \) in equation (8). When there is not enough disagreement, demand curves flip in the “wrong” direction: If the price increases, then each trader thinks that other traders had positive information and therefore trades in the same direction, rather than being confident enough in his own signal to trade against others.
Equilibrium Properties. The equilibrium price is the average of all trader’s valuations of a risky asset. It is equal to the weighted average of the public signal $i_0$ with precision $\tau_0$ and $N$ private signals $i_n$ with effective precision $(\tau_H + (N - 1)\tau_L)/N$ each. The equilibrium price is also equal to the expected value of fundamentals under the beliefs of a representative agent who assigns precision $\tau_0$ to a public signal and precision $(\tau_H + (N - 1)\tau_L)/N$ to each of $N$ private signals. There is no risk adjustment, because a risky asset is in zero-net supply and there are no noise traders.

Under the assumption that all private signals have the same precision, the price is fully revealing. In relative overconfidence case, an economist concludes that market prices are set efficiently, there are no profitable opportunities, and auto-covariance between stock return on date 0 and date 1 is zero. In absolute overconfidence, an economist concludes that market prices overreact in response to information available about fundamentals, and there is negative auto-covariance between stock return on date 0 and date 1.

Each trader thinks that the price incorporates his signal incorrectly, because other traders assign a much lower precision to his private signal. He also thinks that the price incorporates private signals of others incorrectly, because other traders assign a too high precision to their own signals. In equilibrium, each trader trades on his disagreement with the market, taking into account his monopoly power with regard to his “superb” understanding of a true value of signals. The trade $x_n^*$ depends on how much his own signal $i_n$ deviates from signals of others, as inferred from price $p$, and how large his inventory $S_n$ is.

If $S_n^{TI}$ denotes a target inventory of a trader $n$ such that—given this inventory—he does not want to trade, i.e., $x_n^* = 0$,

$$S_n^{TI} = \frac{(N - 1) \cdot (\tau_H - \tau_L)}{A \cdot N} \cdot (\tilde{i}_n - \tilde{i}_{-n}), \quad (17)$$

then his optimal demand can be written as

$$x_n^* = \delta \cdot (S_n^{TI} - S_n), \quad (18)$$

where parameter $\delta$ is defined in equation (14).
A target inventory is based on the difference between trader $n$’s valuation and the valuation of other traders. The absolute value of trader $n$’s target inventory increases in the disagreement $|\hat{i}_n - \tilde{i}_{-n}|$, increases with overconfidence $\tau_H - \tau_L$ and decreases in the risk aversion $A$. Even if $N$ goes to infinity, traders continue to agree to disagree and their target inventories are not zero. The parameter $\delta$ determines the speed with which traders adjust positions towards target inventory levels. Since $\delta < 1$, traders do not move all the way from initial inventory $S_n$ to target inventory $S_n^{TI}$, their demand is subdued by their market power.

From (12) and (13), we find the coefficients of $x_n$ and $S_n$ in the price impact function $P(x_n, S_n) = \lambda_0 + \lambda_x \cdot x_n + \lambda_S \cdot S_n$ in equation (8),

$$\lambda_x := \frac{1}{(N - 1)\gamma} = \frac{A(\tau_H + (N - 1)\tau_L)}{((N - 2)\tau_H - 2(N - 1)\tau_L) \bar{\tau}},$$

(19)

and

$$\lambda_S := \frac{\delta}{(N - 1)\gamma} = \frac{A(\tau_H + (N - 1)\tau_L)}{(N - 1)(\tau_H - \tau_L) \bar{\tau}}.$$  

(20)

The price impact is lower when traders become more confident (fixing $\tau_H + (N - 1)\tau_L$ while increasing $\tau_H$) or competition becomes more intensive (fixing total precision $\bar{\tau}$ while increasing $N$). Intuitively, traders are more willing to trade with others and thus provide more liquidity to the market.

When there is not enough disagreement to sustain an equilibrium with pure strategies, it is possible there is an equilibrium in randomized strategies. For randomized strategies to be an equilibrium, the trader must be indifferent across the various choices of quantities he trades. If we add normally distributed noise symmetrically across all traders, a randomized equilibrium requires the second order condition to be exactly zero. This means that the quadratic objective function reduces to a linear function, i.e., the denominator in the equation for optimal quantity traded is zero. Since the trader has to be indifferent across various randomizations, this further implies that the linear function must be a constant, independently of the quantity traded. This assumption can not hold, because a trader with a positive value of $i_n$ would always want to buy unlimited quantities and a trader with a negative $i_n$ would always want to sell unlimited quantities. This proves that
equilibrium with symmetric normally distributed noise cannot exist. When noise is not normally distributed or the equilibrium is not symmetric, the objective is not quadratic any more, but it will still be difficult to find a mixed strategy equilibrium, given that the sensitivity of utility to a the trader’s own private information must be well-defined.

We will see next that in the dynamic continuous-time setting, the equilibrium price remains fully revealing, but a representative agent does not exist and everyone thinks that market prices are inefficient, even in a relative overconfidence case. Traders continue to subdue their trading, and a partial adjustment process governs the rate at which traders move from their current inventory towards their target inventory.

II. Continuous-time Model

Suppose there are $N$ risk-averse oligopolistic traders, where $n = 1, \ldots, N$, who trade a risky asset with a zero net supply against a risk-free asset. Let $r$ denote a risk-free rate of interest. At time $t$, each trader has inventory $S_t$ of a risky asset. He chooses the consumption $c_t$ and the trading intensity $x_t$ with which he will buy or sell a risky asset to maximize his expected constant absolute risk aversion (CARA) utility function,

$$
\max_{\{c_t, x_t\}} E \left[ \int_{s=t}^{\infty} -e^{-\rho(s-t)} \cdot U(c_s) \cdot ds \right],
$$

where $U(c_s) = -e^{-Ac_s}$ with a risk aversion parameter $A$ and a time-preference parameter $\rho$. Since the model is the same from the perspective of each trader, we consider the optimization problem from the perspective of trader $n$ and drop a subscript $n$ for convenience.

There is also an economist who is studying the properties of the market, where oligopolistic traders trade based on their disagreement about asset prices.

**Bayesian Updating.** Suppose a risky asset continuously pays out dividends $D(t)$ and self-liquidates itself over time. The fundamental value of a risky asset is the expected present value of all future dividends discounted at a risk-free rate $r$. Dividends follow
a stochastic process with the mean-reverting growth rate $G^*(t)$, constant instantaneous volatility $\sigma_D > 0$, and constant rate of mean reversion $\alpha_D > 0$,

$$ dD(t) = -\alpha_D \cdot D(t) \cdot dt + G^*(t) \cdot dt + \sigma_D \cdot dB_D, \quad (22) $$

$$ dG^*(t) = -\alpha_G \cdot G^*(t) \cdot dt + \sigma_G \cdot dB_G. \quad (23) $$

Dividends $D(t)$ are publicly observable. Growth rate $G^*(t)$ is not publicly observable, and traders may argue about it. Defining $dI_0(t) = [\alpha_D \cdot D(t) \cdot dt + dD(t)] / \sigma_D$, $\tau_0 = \sigma_G^2 / \sigma_D^2$, and $dB_0 = dB_D$, we can write the public information $I_0$ in the divided stream (22) as,

$$ dI_0(t) = \tau_0^{1/2} \cdot \frac{G^*(t)}{\sigma_G} \cdot dt + dB_0. \quad (24) $$

Each trader $n$ is also endowed with the stream of private information $I_n(t)$ about the unobserved growth rate $G^*(t)$,

$$ dI_n(t) = \tau_n^{1/2} \cdot \frac{G^*(t)}{\sigma_G} \cdot dt + dB_n, \quad n = 1, \ldots, N, \quad (25) $$

where $dB_D, dB_G, dB_1, \ldots dB_N$ are independent.\(^1\) Information $I_n(t)$ is re-scaled so that the variance of the Brownian motion part is normalized to one. The variance of information flow, which can be calculated infinitely precisely from past signals, does not depend on the precision of information flow and does not provide any reason for traders to stop disagreeing with each other. In some sense, the signal $dI_n(t)$ is information about a “Sharpe ratio”, $G^*(t) / \sigma_G$.

Trader $n$ observes public signal $I_0(t)$ and private signal $I_n(t)$. The symmetric equilibrium also reveals the average of other traders’ signals defined by $I_{-n}(t) := \frac{1}{N-1} \sum_{m \neq n} I_m(t)$. Traders agree about the precision of the public signal $\tau_0$, but disagree about the precisions of private signals. Trader $n$ believes that his signal $I_n(t)$ has precision of $\tau_n = \tau_H$, whereas the signals $I_m(t)$ of the other traders, $m \neq n$, have precision $\tau_m = \tau_L$, where $0 < \tau_L < \tau_H$. Traders agree about the total precision is $\bar{\tau} = \tau_0 + \tau_H + (N-1) \cdot \tau_L$.

\(^1\)As in the “Gennotte Notes,” we can specify a negative correlation between $dB_D$ and $dB_G$ such that $E\{G^*(t)|D(u), u < t\} = 0$. This does not seem to change the main results.
Let us assume that an economist embarks on studying properties of securities prices in that market. He is interested in whether market prices are set efficiently or it is possible to find anomalies. Economist assigns the precision $\tau_0$ to the stream of public information flow $dI_0$ in dividends process and the same precisions of $\tau_e$ to the information flows $dI_1, \ldots, dI_N$. Denote $\tau^E = \tau_0 + N \cdot \tau_e$ being the precision of all information available in the system according to an economist.

Traders “agree to disagree” about the precision of signals. This disagreement is a “common knowledge,” and it makes traders trade in equilibrium. Without overconfidence—i.e. in a model with rational expectations—there would be no trade in equilibrium. The perceived precisions $\tau_L$ and $\tau_H$ affect the equilibrium prices and quantities. In contrast, the “true precisions” of an economist have no affect on the equilibrium, but change its interpretation.

**Lemma 1** Denote $\tau := \sum_{n=0}^{N} \tau_n$. Let $G(t)$ be the estimate of a true growth rate $G^*(t)$ given history of signals $D, I_1, I_2, \ldots I_N$ with generic beliefs $\tau_0, \tau_1, \ldots \tau_N$ and the error variance $\Omega := Var\left[\frac{G(t)}{\sigma_G} - \frac{G^*(t)}{\sigma_G}\right]$. Then,

$$\Omega = \frac{\sqrt{\alpha^2_G + \tau} - \alpha_G}{\tau}$$

$$dG(t) = - (\alpha_G + \Omega \cdot \tau) \cdot G(t) \cdot dt + \sigma_G \cdot \Omega \cdot \sum_{n=0}^{N} \tau_n^{1/2} \cdot dI_n. \quad (27)$$

The error variance $\Omega$ corresponds to a steady state that balances an increase in error variance due to stochastic change $dB_G$ in a true growth rate with a reduction in error variance due to a mean-reversion of a true growth rate and arrival of new information about it. The dynamics of the estimate $G(t)$ consists of two adjustments: The new estimate has to be updated because the old estimate has changed upon arrival of new information $\left(dI_n - \tau_n^{1/2} \cdot \sigma^{-1}_G G(t) \cdot dt\right)$ and a true growth rate $G^*(t)$ itself has changed.

From equation (27), the estimate $G(t)$ can be conveniently written as the weighted
sum of $N + 1$ sufficient statistics $H_n$ corresponding to information flow $dI_n$,

$$G(t) = \sigma_G \cdot \Omega \cdot \sum_{n=0}^{N} \tau_n^{1/2} \cdot H_n(t),$$

(28)

where

$$H_n(t) := \int_{u=-\infty}^{t} e^{-(\alpha_G + \Omega \tau) \cdot (t-u)} \cdot dI_n(u), \quad n = 0, 1, \ldots, N,$$  

(29)

d$$H_n(t) = -((\alpha_G + \Omega \tau) \cdot H_n(t) \cdot dt + dI_n(t), \quad n = 0, 1, \ldots, N.$$

(30)

The importance of each bit of information $dI_n$ about a growth rate $G(t)$ decays exponentially at a rate $\alpha_G + \Omega \cdot \tau$, being the same across traders.

Note that equations (24), (25) and (27) imply that the estimate $G(t)$ mean-reverts to zero at a rate $\alpha_G$,

$$dG(t) = -\alpha_G \cdot G(t) \cdot dt + \Omega \tau (G^* - G) \cdot dt + \sigma_G \cdot \Omega \cdot \sum_{n=0}^{N} \tau_n^{1/2} \cdot dB_n(t).$$

(31)

Each trader $n$ has beliefs $\tau_0, \tau_n = \tau_H$, $\tau_m = \tau_L$ for $m \neq n$ and $m \neq 0$ such that $\bar{\tau} = \tau_0 + \tau_H + (N-1)\tau_L$. Traders agree on the total precision $\bar{\Omega}$ in the market. Consequently, all of them calculate the error variance $\bar{\Omega}$ by plugging $\bar{\tau}$ instead of $\tau$ into equation (26). They also agree that the correct way to process available information is to construct signals $H_n(t), n = 0, \ldots, N$ by plugging $\bar{\tau}$ and $\bar{\Omega}$ instead of $\tau$ and $\Omega$ into equation (29). Traders disagree, however, on how to aggregate signals $H_n(t), n = 0, \ldots, N$ into their estimate of a growth rate in equation (28) and choose to assign a bigger weight to their own signal relative to others. We define trade $n$’s estimate of a true growth rate $G_n(t)$ corresponding to beliefs $\tau_0, \tau_n = \tau_H$, $\tau_m = \tau_L$ for $m \neq n$ and $m \neq 0$ as

$$G_n(t) := \sigma_G \cdot \bar{\Omega} \cdot \left( \tau_0^{1/2} \cdot H_0(t) + \tau_H^{1/2} \cdot H_n(t) + (N-1)\tau_L^{1/2} \cdot H_{-n}(t) \right),$$

(32)

where

$$H_{-n}(t) := \frac{1}{N-1} \sum_{m=1, m \neq n}^{N} H_m(t),$$

(33)
Trader $n$’s estimate of dividend growth rate can be also written as

$$G_n(t) = \sigma_G \cdot \tilde{\Omega} \cdot \tau_H^{1/2} \cdot \dot{H}_n(t) + \sigma_G \cdot \tilde{\Omega} \cdot (N-1) \tau_L^{1/2} \cdot \dot{H}_{-n}(t).$$  

(34)

where $\hat{A} := \tau_0^{1/2} \cdot \left( \tau_H^{1/2} + (N-1)\tau_L^{1/2} \right)^{-1}$, $\dot{H}_n(t) := H_n(t) + \hat{A} \cdot H_0(t)$ and $\dot{H}_{-n}(t) := H_{-n}(t) + \hat{A} \cdot H_0(t)$.

From equations (25), (30), and (33), we derive the dynamics of sufficient statistics $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$ and present it in Appendix. Their dynamics is complicated. For example, the dynamics of $H_n(t)$ in equation (30) depends on the mean-reversion and new information $dI_n(t)$, which in turn has a predictable part $\tau_n^{1/2} \sigma_G^{-1} \cdot G(t) \cdot dt$, where $G(t)$ is a complicated mixture of $H_0(t)$, $H_n(t)$ and $H_{-n}(t)$ with weights proportional to $\tau_0^{1/2}$, $\tau_H^{1/2}$ and $(N-1)\tau_L^{1/2}$, as seen from equation (32).

**Economist’s Beliefs.** An economist forms an estimate of a dividend growth rate from the history of dividends and the history of average private signals $\bar{H}(t) := \frac{1}{N} \sum_{n=1}^{N} H_n(t)$, which can be inferred—as we show later—from the history of prices.

Suppose an economist believes that the total amount of information is equal to $\tau^E$, and so his error variance $\Omega^E = (\sqrt{\alpha_G^2 + \tau^E} - \alpha_G)/\tau^E$ from equation (26). Since $\tau^E$ and $\Omega^E$ may differ from $\bar{\tau}$ and $\bar{\Omega}$, he may disagree with traders on how to use the same flow of information. Instead of signals $H_n(t), n = 0, \ldots, N$, he will construct an alternative set of signals $K_n(t), n = 0, \ldots, N$ by plugging $\tau^E$ and $\Omega^E$ into equation (29). Both sets of signals represent a weighted sum over the same information flow, but an economist believes that information decays with a rate of $\alpha_G + \Omega^E \tau^E$ rather than $\alpha_G + \bar{\Omega} \bar{\tau}$. The equation (30) for a pair $\tau^E$ and $\Omega^E$ as well as a pair $\bar{\tau}$ and $\bar{\Omega}$ yields a simple relation between both sets of signals, $n = 0, 1, \ldots, N$, $dK_n(t) - dH_n(t) = - (\alpha_G + \Omega^E \tau^E) \cdot (K_n(t) - H_n(t)) \cdot dt + (\bar{\Omega} \bar{\tau} - \Omega^E \tau^E) \cdot H_n(t) \cdot dt$. Using this equation, an economist can construct his own signals $K_n(t), n = 0, \ldots, N$ from signals $H_n(t), n = 0, \ldots, N$ and calculate his estimate of a dividend growth rate $G^E(t)$,

$$K_n(t) = H_n(t) + Z^E_n(t),$$

(35)

$$Z^E_n(t) = (\bar{\Omega} \bar{\tau} - \Omega^E \tau^E) \cdot \int_{u=-\infty}^{t} e^{-(\alpha_G + \Omega^E \tau^E) \cdot (t-u)} \cdot H_n(u) \cdot du,$$

(36)
\[ G^E(t) := \sigma_G \cdot \Omega^E \cdot \left( \tau_0^{1/2} \cdot (H_0(t) + Z_0^E(t)) + \tau_e^{1/2} \cdot N \cdot (\bar{H}(t) + \bar{Z}^E(t)) \right). \] (37)

The terms \( Z_0^E(t) \) and \( \bar{Z}^E(t) := \frac{1}{N} \sum_{n=1}^{N} Z_n^E(t) \) are complicated averages of the past data. These are adjustments that an economist makes in order to reinterpret past and current prices, given his disagreement with traders about how much precision is contained in information flow. The estimate \( G^E \) depends in a complicated manner not only on the last realization of dividends \( D(t) \) and average signal \( \bar{H}(t) \) but also on their entire history up to time \( t \).

The inference problem is greatly simplified in the case of relative overconfidence, when traders agree with each other and with an economist on what the total precision, \( \bar{\tau} = \tau^E \). An economist agrees with traders on how quickly information becomes obsolete and how to update signals when new information arrives. He uses the same signals as all traders, \( H_n, n = 0, 1, \ldots N \), without looking at the past data—\( Z_n^E(t) = 0, n = 0, \ldots N \)—but assigns different weights to that signals when forming his estimate of a dividend growth rate (37). In other words, an economist’s estimate depends only on the last realization of the dividend and the equilibrium price.

In the case of absolute overconfidence, traders think that there is more information in the market than the economist, \( \bar{\tau} > \tau^E \). They use Kalman filtering to learn from their own flow of signals and the history of prices. An economist with correct beliefs would prefer to average past prices differently from how traders do it. This makes an economist’s inference problem complicated.

**Utility Maximization with Market Power.** There are five state variables: money market \( M(t) \) in dollars, inventory \( S(t) \) in shares, dividend \( D(t) \) in dollars, and two variables describing changing growth rate of dividends, \( \dot{H}_n(t) \) and \( \dot{H}_{-n}(t) \). Trader \( n \) maximizes his utility function over consumption plan \( c_t \) and trading rate \( x_t \). He explicitly takes into account the effect of his trading on the price of a risky asset \( P(x) \). Let \( V(M, S, D, \dot{H}_n, \dot{H}_{-n}) \) be the value function at time \( t \) for the optimal consumption and
investment policy \((c, x)\),

\[
V(M, S, D, \hat{H}_n, \hat{H}_{-n}) := \max_{\{c_t, x_t\}} E_t \left[ \int_{s=t}^{\infty} e^{-\rho(s-t)} \cdot e^{-A_c s} \cdot ds \right],
\tag{38}
\]

where the state variables satisfy stochastic differential equations

\[
dM(t) = (r \cdot M(t) + S(t) \cdot D(t) - c_t - P(x_t) \cdot x_t) \cdot dt,
\tag{39}
\]

\[
dS(t) = x_t \cdot dt,
\tag{40}
\]

\[
dD(t) = -\alpha_D \cdot D(t) \cdot dt + G_n(t) \cdot dt + \sigma_D \cdot dB_D + (G^*(t) - G_n(t)) \cdot dt,
\tag{41}
\]

and state variables \(\hat{H}_n(t)\) and \(\hat{H}_{-n}(t)\) follow dynamics described in (83) and (84) in the Appendix. In addition, \(V(M(t), S(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t))\) satisfies the transversality condition

\[
\lim_{t \to +\infty} E[e^{-\rho t}V(M(t), S(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t))] = 0.
\tag{42}
\]

We conjecture that traders smooth out their trading. The trajectories of their inventories \(S(t)\) are differentiable. Infinitely fast portfolio updating cannot be an equilibrium. Indeed, each trader would then believe that he could lower his execution costs by trading more slowly than the other traders—essentially by walking up or down the residual demand schedules they present to him—but all traders can not trade more slowly than average. In our model, we specify trading strategies and price impact functions in terms of rates of trading \(x_t\), not shares traded. This is a key point.

**Linear Conjectured Strategies.** Based on public information, including the history of market clearing prices, and their private information, each trader submits a demand schedule for the rate at which he will buy the asset during period \([t, t+dt)\) as a function of the market clearing price. An auctioneer establishes a market clearing price. In particular, there is the following sequence of events. First, trader \(n\) observes \(D(t + \Delta t)\) and \(\hat{H}_n(t + \Delta t)\). Second, he submits demand schedule for a rate of trading \(x_n(t + \Delta t) = X_n(D + \Delta D, \hat{H}_n + \Delta \hat{H}_n, S(t), P(t + \Delta t))\) to an auctioneer. Third, he learns about realized market-
clearing equilibrium price $P(t + \Delta t)$ and his realized trading rate $x_t = X_n(D + \Delta D, \hat{H}_n + \Delta \hat{H}_n, S(t), P(t + \Delta t))$. Fourth, he infers $\hat{H}_{-n}(t + \Delta t)$ from equilibrium price $P(t + \Delta t)$. Although the price is fully revealing, the traders agree to disagree and continue trading on their information in the equilibrium.

Trader $n$ conjectures that the other $N - 1$ traders submit symmetric linear demand schedules for rates of trading, $m \neq n$,

$$X_m(t + \Delta t) = \gamma_D \cdot D(t + \Delta t) + \gamma_H \cdot \hat{H}_m(t + \Delta t) - \gamma_S \cdot S_m(t) - \gamma_P \cdot P(t + \Delta t), \quad (43)$$

where constants $\gamma_D$, $\gamma_H$, $\gamma_S$, and $\gamma_P$ will be determined in equilibrium. These constants are known to each trader.

From the market clearing condition and the linear specification of demand schedules for traders, it follows that

$$x_t + \sum_{m \neq n}(\gamma_D \cdot D(t + \Delta t) + \gamma_H \cdot \hat{H}_m(t + \Delta t) - \gamma_S \cdot S_m(t) - \gamma_P \cdot P(t + \Delta t)) = 0. \quad (44)$$

Differently from Kyle(1989), the market-clearing condition is specified in terms of trading rates rather than positions that traders establish. Since $\sum_{m=1}^{N} S_m = 0$, solving for $P$ as a function of $x_t$ yields the following price impact functions that trader $n$ faces:

$$P(x_t) = \frac{\gamma_D}{\gamma_P} \cdot D(t + \Delta t) + \frac{\gamma_H}{\gamma_P} \cdot \hat{H}_{-n}(t + \Delta t) + \frac{\gamma_S}{\gamma_P} \frac{1}{N - 1} \cdot S_n(t) + \frac{1}{(N - 1)\gamma_P} \cdot x_t. \quad (45)$$

The residual demand curve depends on the trader $n$’s rate $x_t$ of trading during period $[t, t + dt)$ rather than the number of shares $x_t \cdot dt$ traded at time $t$. Plugging price impact function (45) into optimization problem (38), trader $n$ determines his optimal consumption and demand schedule. The manner in which trader $n$ exploits the market-clearing rule makes equilibrium concept imperfectly competitive.
Conjectured Value Function. We conjecture that the value function \( V(M, S, D, \hat{H}_n, \hat{H}_{-n}) \) has a following form,

\[
V(M, S, D, \hat{H}_n, \hat{H}_{-n}) = -\exp \left( \psi_0 + \psi_M M + \frac{1}{2} \psi_{SS} S^2 + \psi_{SD} SD + \psi_{Sn} \cdot \hat{S}H_n + \psi_{Sx} \cdot S\hat{H}_n \right.
\]

\[
+ \frac{1}{2} \psi_{nn} \cdot \hat{H}_n^2 + \frac{1}{2} \psi_{xx} \cdot \hat{H}_{-n}^2 + \psi_{nx} \cdot \hat{H}_n \hat{H}_{-n} \biggr),
\]

where nine constants \( \psi_0, \psi_M, \psi_{SS}, \psi_{SD}, \psi_{Sn}, \psi_{Sx}, \psi_{nn}, \psi_{xx}, \) and \( \psi_{nx} \) will be determined in equilibrium. We then write the Hamilton-Jacobi-Bellman (HJB) equation for the value function and solve for the optimal consumption and demand schedule for a rate of trading:

\[
c^*_n(t) = -\frac{1}{A} \left( \log(-\frac{\psi_M}{A}) + \log(-V(t)) \right),
\]

\[
x^*_n(t) = \frac{(N - 1)\gamma_P}{2\psi_M} \left( \left( \frac{\psi_{SS}}{(N - 1)\gamma_P} \cdot S(t) + \psi_{Sn} \cdot \hat{H}_n + \left( \frac{\psi_{Sx}}{\gamma_P} \right) \cdot \hat{H}_{-n}(t) \right) \right).
\]

To obtain the solution, we made a conjecture that the optimal trading strategy does not depend on publicly observed dividend \( D(t) \), i.e., \( \gamma_D = \gamma_P \cdot \psi_{SD}/\psi_M \). This conjecture holds in equilibrium.

This optimal trading strategy is given by a linear combination of observable \( S(t) \), \( \hat{H}_n \), \( P \) and unobservable \( \hat{H}_{-n} \), but it can be also implemented as a linear function of observable \( D(t), S(t), \hat{H}_n \) and \( P \). Given the aforementioned conjecture about \( \gamma_D \), trader \( n \) can infer \( \hat{H}_{-n} \) from the market-clearing condition (44) as,

\[
\hat{H}_{-n} = \frac{\gamma_P}{\gamma_H} \left( P - D \cdot \frac{\psi_{SD}}{\psi_M} \right) - \frac{1}{(N - 1)\gamma_H} \cdot x^* - \frac{\gamma_S}{(N - 1)\gamma_H} \cdot S.
\]

Plugging (49) into equation (48) and solving for \( x^* \) yields the presentation of trading strategy \( x^* \) as a linear function of \( \hat{H}_n, S(t) \), and \( P - D \cdot \frac{\psi_{SD}}{\psi_M} \), observable for trader \( n \).

In symmetric equilibrium, coefficients in this linear function should coincide with coefficient in the conjectured strategy for other traders \( x = \gamma_H \cdot \hat{H}_n - \gamma_S \cdot S - \gamma_P \cdot \left( P - D \cdot \frac{\psi_{SD}}{\psi_M} \right) \).

Equating corresponding coefficients gives us three equations, solving which for \( \psi_{Sx}, \gamma_H, \)
Substituting (50) into equation (48) yields the solution for optimal strategy. Plugging (47) and (48) back into the HJB equation and setting the constant term, coefficients of $M$, $SD$, $S^2$, $SH_n$, $SH_{-n}$, $H_n^2$, $H_{-n}^2$ and $\hat{H}_n\hat{H}_{-n}$ to be zero, we get the other nine equations. In total, there are nine equations with nine unknowns $\gamma_P$, $\psi_0$, $\psi_M$, $\psi_{SD}$, $\psi_{SS}$, $\psi_{Sn}$, $\psi_{nn}$, $\psi_{xx}$, and $\psi_{nx}$. Solving this system yields the solution. Define

$$G := \frac{1}{N} \sum_{n=1}^{N} G_n,$$  $(51)$

$$C_L := -\frac{\psi_{Sn}}{2\psi_{SS}}, \quad C_H := \frac{(N-1)\gamma_P\psi_{Sn}}{2\psi_M}, \quad C_G := \frac{\psi_{Sn} \hat{A}N(r + \alpha_D)(r + \alpha_G)}{2\psi_M \sigma_G \Omega \sqrt{\theta}}.$$  $(52)$

**Theorem 2** In a symmetric linear flow equilibrium with “smooth trading” of the form $dS_n = x_n \cdot dt$, trader $n$’s value function $V(M, S, D, \hat{H}_n, \hat{H}_{-n})$ is given in equation (46).

1. Trader $n$’s optimal consumption $c_n^*(t)$ is given in (47) and demand schedule for a rate of trading $x_n^*(t)$ is

$$x_n^*(t) = \gamma_S \cdot \left( C_L \cdot (\hat{H}_n(t) - \hat{H}_{-n}(t)) - S_n(t) \right).$$  $(53)$

2. The equilibrium price is

$$P^*(t) = \frac{D(t)}{r + \alpha_D} + \frac{C_G \cdot \hat{G}(t)}{(r + \alpha_D)(r + \alpha_G)},$$  $(54)$

where $\psi_M = -rA < 0$, $\psi_{SD} = -rA/(r + \alpha_D) < 0$, and $\psi_0 < 0$ is given explicitly in equation (89). The constants $\gamma_H$, $\gamma_S$, and $\psi_{Sx}$ are given in equations (50). The six constants $\gamma_P > 0$, $\psi_{SS} > 0$, $\psi_{Sn} < 0$, $\psi_{nn}, \psi_{xx}$, and $\psi_{nx}$ are determined from the system of six polynomial equations (90)-(95) in the Appendix.
The symmetric linear equilibrium of the continuous-time model shares many features of the equilibrium of one-period model. The equilibrium price \( P^*(t) \) has a form similar—but not exactly the same—to the Gordon’s formula for an asset with the current dividend \( D(t) \) and the estimate of a growth rate \( \bar{G}(t) \), equal to the average of traders’ estimates. The equilibrium price (54) can be also written as a function of dividend \( D(t) \), signal \( H_0(t) \), and the average of private signals \( \bar{H}(t) = \frac{1}{N} \sum_{i=1}^{N} H_i(t) \),

\[
P^*(t) = \frac{D(t)}{r + \alpha_D} + \frac{C_G \cdot \sigma_G \cdot \bar{\Omega}}{(r + \alpha_D)(r + \alpha_G)} \cdot \left( \tau^{1/2}_0 \cdot H_0(t) + N \cdot \tau^{1/2} \cdot \bar{H}(t) \right).
\]

(55)

where \( \tau^{1/2} := (\tau_H^{1/2} + (N-1) \cdot \tau_L^{1/2})/N \). Note that the equilibrium price reveals the average of private signals. An economist with symmetric beliefs thinks that traders are wrong, because each trader assigns a bigger weight to his own signal, but a true estimate of a dividend growth rate has to be an equally-weighted sum of private signals. By inferring an average private signal from equilibrium prices and readjusting it, an economist will be able to construct his estimate of a growth rate.

The equilibrium trading strategies have a simple form similar to those in one-period model. Let \( S_{TI}^n(t) \) be defined as the target demand of trader \( n \), when he does not want to trade and optimally chooses trading intensity \( x_n^*(t) = 0 \),

\[
S_{TI}^n(t) = C_L \cdot (\hat{H}_n(t) - \bar{H}(t)).
\]

(56)

The optimal trading rate of trader \( n \) is

\[
x_n^*(t) = \gamma_S \cdot (S_{TI}^n(t) - S_n(t)).
\]

(57)

The absolute value of target inventories \( |S_{TI}^n(t)| \) increases with disagreement \( |\hat{H}_n(t) - \bar{H}_n(t)| \). As in one-period model, traders only partially move from their current inventory \( S_n(t) \) towards their target inventory \( S_{TI}^n(t) \). Since \( dS_n(t) = x_n^*(t) \cdot dt \), equation (57) implies that trader \( n \) expects his inventory \( S_n(t) \) converge exponentially over time to the target inventory at a speed \( \gamma_S \). There are no jumps in their optimal strategies. Traders expect to smooth out their trading.
As in one-period model, there is always a no-trade equilibrium, in which the market price is not defined. If each trader submits a demand schedule \( x_n(t) = 0 \), then such a no-trade demand schedule is optimal for all traders. This is not a symmetric linear equilibrium, because the auctioneer cannot establish a meaningful market price.

### III. Analysis of the Equilibrium

#### A. Trades, Prices, and Fundamentals.

Prices adjust quickly: Equilibrium prices immediately and fully reveal average signal of all traders. In contrast, quantities adjust slowly: Trading on information innovations continues even after signals are revealed in prices.

When a trader observes a new signal, he updates his estimate of the growth rate, recalculates his target inventory, and immediately adjusts the rate of trading towards the new target. As soon as a trader changes the speed of trading, the price of a risky asset instantaneously jumps to a new equilibrium level, even though a trader has not traded yet. Since block trades are infinitely expensive, a trader does not trade immediately to the new target, but instead adjusts his inventories slowly taking into account his market power and information flow.

If a trader follows the equilibrium strategy and trades at a rate \( x^*_n(t) = \gamma_S \cdot (S^T(t) - S_n^*(t)) \) towards his target inventories \( S^T(t) = C_L \cdot (H_n(t) - H_{-n}(t)) \), then his inventory \( dS^*_n(t) = x^*_n(t) \cdot dt \) evolves as

\[
S^*_n(t + T) = e^{-\gamma_s T} \cdot \left( S^*_n(t) + \int_{t}^{t+T} e^{-\gamma_s (t-u)} \cdot \gamma_S \cdot C_L \cdot (H_n(u) - H_{-n}(u)) \cdot du \right). \tag{58}
\]

Each moment traders gradually liquidate current inventories and also adjust their targets in response to a new information. This realistic modeling of inventory management is an important feature of our model. Even if disagreement \( H_n(u) - H_{-n}(u) \) between traders does not change, traders continue to trade based on their “past” disagreement. Plugging the optimal strategy \( x^*_n(t) \) from equation (53) into equation (45) yields the equilibrium
price $P^*$ in equation (54), which does not change as long as disagreement remains the same. As a trader continues to trade towards the target inventory, the price does not change, except when new information arrives.

The equilibrium prices differ from fundamental prices. Let $F_n(t)$ denote the fundamental price of an asset under beliefs of trader $n$. Each trader thinks that his estimate of a dividend growth rate mean-reverts to zero at a rate $\alpha_G$. This implies that the fundamental value is determined by the Gordon’s formula with the current dividend $D(t)$ and the growth rate $G_n(t)$,

$$F_n(t) = D(t) + \frac{G_n(t)}{(r + \alpha_D)(r + \alpha_G)}.$$  \hspace{1cm} (59)

Traders usually disagree with each other and the equilibrium price, because their estimates of a dividend growth rate usually differ from the estimates of others and their average.

The dynamic model has an interesting property. Even if traders happen to agree on the level of current dividend growth rate, they will still find that the market price depart from the market consensus of the fundamental value and there are profitable opportunities. Indeed, our numerical analysis shows that the coefficient $C_G$ in equation (54) is always less than one. This implies that even when $G_n = \bar{G}$ for all $n = 1,...,N$, the equilibrium price (54) and fundamentals (59) are not equal to each other. We refer to $C_G < 1$ as “dampening effect.”

Figure 1 illustrates the intuition. Suppose all traders happened to agree that the level of current dividend growth rate is equal to $G_n(t) = \bar{G}(t)$. Since all traders think that their own estimate $G_n(t)$ mean-reverts to zero at a rate $\alpha_G$, they agree on the fundamental price $F(t) = \frac{D(t)}{r + \alpha_D} + \frac{\bar{G}}{(r + \alpha_D)(r + \alpha_G)}$. Under the assumption that no new information arrives, the projected dynamics of fundamentals is depicted in figure 1 in dark blue for two symmetric cases, when $G$ is positive and when $G$ is negative.

Despite their consensus, the equilibrium price $P^*$ will be lower than fundamentals if $F > 0$ and higher than fundamentals if $F < 0$. Each trader thinks that other traders put too much weight on their private signals when forming their estimates and so they happened to agree on the same estimate of a growth rate for a wrong reason. Each trader $n$ thinks that other traders will soon find out that their estimates mean-revert to zero.
Figure 1: Figure shows the projected dynamics of fundamental value according to each trader in dark blue, the projected dynamics of other traders’ estimates of fundamental value according to each trader in light blue, and the resulting equilibrium prices in red. All traders agree on fundamental value $F$ and its dynamics, but the equilibrium price $P^*$ differs from $F$.

faster than at a rate $\alpha_G$, because it is only component $H_n$ that mean-reverts to zero at a rate $\alpha_G$ whereas other signals $H_{-n}$ mean-revert to zero faster. As a result, everybody expects that estimates of others will first deviate towards zero and then converge to the commonly agreed upon fundamental value $F$, as depicted on figure 1 in light blue. Since everybody knows that they will be able to trade at more favorable prices in the nearest future, they do not agree to trade at fundamental price $F$. This makes the equilibrium price “dampened” relative to fundamentals. Everybody agrees, however, that as the time passes, the price will converge to their estimates of fundamentals, as depicted on figure 1 in red.

Our model bears a resemblance to the description of financial markets by Keynes (1936). Keynes wrote that stock prices are usually governed by average expectations of those who deal on the exchange, most of whom are not planning to hold securities for a long time and therefore not thinking about the state of long-term expectations. Instead, their energies and skills are occupied with forecasting the short-term price changes. “For
most of these persons are, in fact, largely concerned, not with making superior long-term forecasts of the probable yield on an investment over its whole life, but with foreseeing changes in the conventional basis of valuation a short time ahead of the general public. They are concerned, not with what an investment is really worth to a man who buys it “for keeps,” but with what the market will value it at, under the influence of mass psychology, three months or a year hence.”

As in Keynes (1936), traders in our model appear to be preoccupied with the “short-term price dynamics” rather than “hold-to-maturity” values. Even though all of them are aware of profit opportunities—since the market prices differ from their long-term expectations about fundamentals—they choose to forego these opportunities because of their particular expectations about the short run dynamics. As Keynes puts it, “it is not sensible to pay 25 for an investment of which you believe the prospective yield to justify value of 30, if you also believe that the market will value it at 20 three months hence.”

Keynes also believed that since financial markets are dominated by short-term speculation rather than long-term enterprise, they are not too different from a casino. This is the outcome of the particular organization of the financial markets where ownership is separated from management.

In contrast to Keynes (1936), prices adjust instantly, even though traders trade only slowly towards their long-term targets. The market prices are fully revealing. In the equilibrium, the beauty-contest phenomenon of Keynes (1936) is internalized. The market prices are less sensitive to changes in the average estimate of dividend growth rate. Instead of excessive volatility forecasted by Keynes, the disagreement actually dampens price volatility in our model. This result is consistent with Allen, Morris and Shin (2006), who also find that prices in a beauty contest exhibit inertia and react sluggishly to changes in the fundamental value.

B. Information Aggregation and Representative Agent.

One of the fundamental questions in economics is the question about information aggregation. Hayek (1945) writes that the key problem for designing a rational economic
order is that constantly changing information, necessary for making decisions, is dispersed among separate individuals, rather than readily available to a central planner. He further suggested that even if people act in their own interests, the price system will communicate information to others and help to bring about the outcome, which “might have been arrived at by one single mind possessing all the information which is in fact dispersed among all the people involved in the process.”

Adopting this idea, modern economists often hope to gain insights about how economic systems work by relying on the device of a “representative agent.” Representative agent models such as classical work by Lucas (1978) and Mehra and Prescott (1985) are models, in which all agents act in such a manner that their cumulative actions might as well be the actions of one agent maximizing his expected utility function. The representative-agent assumption allows economists to focus on substantial properties of the economy, especially when researchers are interested in aggregate results, instead of carrying along numerous parameters describing agents in the models. Even though the representative agent assumption greatly simplifies economic analysis and brings tractability, it may potentially over-simplify the models and leave out key features of economics systems.

One concern is that representative agent models usually do not have good aggregation properties and fail to provide rigorously derived micro-foundations from “first principles.” Even though the representative agent models greatly advanced the field of economics relative to the earlier studies of relations between statistical aggregates launched by Keynes, the detailed modeling of interaction between individual agents may be still necessary for generating further insights.

Our model provides a perfect illustration. In our model, the price is fully revealing, but a representative agent does not exist. The equilibrium price does not converge to the expected value of dividends conditioning on the pooled traders’ information. It would be misleading to make inferences even about aggregate quantities without thinking about specific interactions between individual traders in a dynamic game-theoretic context.

There are no beliefs of an economist such that his estimate of fundamental value coincides with the market prices. Regardless of his beliefs, an economist always finds that prices deviate from fundamentals and the market is intrinsically inefficient. Our model
provides a guidance on what determines the dynamics of anomalies.

An economist uses the history of dividends and prices to construct his estimate of a growth rate $G^E(t)$. He follows a simple algorithm and first calculates

1. $H_0(u) := \int_{k=-\infty}^{u} e^{-(\alpha_G + \Omega^E_E)\cdot(u-k)} \cdot dI_0(u), \ u \leq t$;

2. $Z_0^E(u) = (\bar{\Omega}\bar{\tau} - \Omega^E_E) \cdot \int_{k=-\infty}^{u} e^{-(\alpha_G + \Omega^E_E)\cdot(u-k)} \cdot H_0(u) \cdot du, \ u \leq t$;

3. $\bar{H}(u)$ from equilibrium prices (55) for all $u \leq t$.

4. $\bar{Z}^E(u) = (\bar{\Omega}\bar{\tau} - \Omega^E_E) \cdot \int_{k=-\infty}^{u} e^{-(\alpha_G + \Omega^E_E)\cdot(u-k)} \cdot \bar{H}(u) \cdot du, \ u \leq t$.

Using these variables, an economist then constructs his estimate of a growth rate as $G^E = \sigma_G^E (\tau_0^{1/2}[H_0 + Z_0^E] + \tau_e^{1/2}N[H + \bar{Z}^E])$. Since he thinks that the estimate mean-reverts to zero at a rate $\alpha_G$, an economist calculates his estimate of a fundamental value as $F^E = \frac{D}{r+\alpha_D} + \frac{\sigma_G^E}{(r+\alpha_D)(r+\alpha_G)} \left( \tau_0^{1/2}[H_0 + Z_0^E] + \tau_e^{1/2}N[H + \bar{Z}^E] \right)$. In the case of relative overconfidence ($\bar{\tau} = \tau^E$ and therefore $\bar{\Omega} = \Omega^E, Z_0^E = 0, \bar{Z}^E = 0$), an economist agrees with traders on how to interpret information and uses only their signals $H_0$ and $\bar{H}$.

A representative agent needs to have beliefs such that the equilibrium price $P^*$ determined by equation (55) and the fundamental price $F^E(t)$ determined in equation above coincide. To align them, the terms $Z_n^E(t), n = 0, \ldots, N$ have to be equal to zero and the coefficients of $H_0(t)$ and $\bar{H}(t)$ in these equations have to be matched,

$$C_G \cdot \bar{\Omega} = \Omega^E \quad \text{and} \quad \tau_e^{1/2} = \hat{\tau}^{1/2} \quad \text{and} \quad \tau^E = \bar{\tau}. \quad (60)$$

The beliefs play two roles. First, they determine the weights with which an economist aggregates new signals into his estimate of a growth rate in equation (28), i.e., $\tau_e^{1/2} = \hat{\tau}^{1/2}$. Second, they determine the speed with which his signal deteriorates and, consequently, price resiliency in equation (30), i.e., $\tau^E = \bar{\tau}$. No symmetric beliefs can simultaneously match both static price level and its dynamics. The last equation requires $\tau_e = (\tau_H + (N-1)\cdot\tau_L)/N$, and this will not never satisfy the second equation $\tau_e^{1/2} = (\tau_H^{1/2} + (N-1)\cdot\tau_L^{1/2})/N$.

It can be shown that even the first two equations are inconsistent with each other.

In other words, there is no representative agent in the dynamic model. This issue does not arise in one-period model, where only static price level has to be matched.
C. Market Efficiency.

Previously discussed issues are closely related to the concept of market efficiency. In a sense of Fama (1970), the market in our model is efficient, because prices reveal all private information. This “efficiency” is yet questionable: An economist and traders disagree with equilibrium prices; everybody thinks that prices do not follow a martingale and it is potentially possible to make money by exploiting existing profitable opportunities.

Suppose \( dF \) is the dynamics of fundamental value \( (59) \) under the generic beliefs that public signal has precision \( \tau_0 \) and private signals have precisions \( \tau_n, \tau = \sum_{n=0}^{N} \tau_n \). Suppose the error variance \( \Omega \) is defined in equation \( (26) \). If prices were to coincide with fundamentals, the return on an investment into a risky asset would be equal to \( dQ_F := [dF] - r \cdot F(t) \cdot dt + D(t) \cdot dt \). Using equations \( (31) \) and \( (59) \), we find

\[
dQ_F = \frac{G^*(t) - G(t)}{r + \alpha_D} \left( 1 + \frac{\Omega r}{r + \alpha_D} \right) dt + \frac{\sigma_D}{r + \alpha_D} dB_D + \frac{\sigma_G \Omega}{(r + \alpha_D)(r + \alpha_G)} \left( \sum_{n=0}^{N} \tau_n^{1/2} dB_n \right). \tag{61}
\]

Since \( G(t) \) is the unbiased estimate of a true growth rate \( G^*(t) \), the drift of this process is equal to zero. Everybody believes that fundamentals follow a random walk. They do not draw the same conclusion about the actual equilibrium returns.

Suppose \( dP^* \) is the dynamics of prices under generic beliefs, then the return on a risky asset is \( dQ_P := dP^* - r \cdot P^*(t) \cdot dt + D(t) \cdot dt \). Plugging equilibrium price \( P^* \) from equation \( (55) \) and using equation \( (25) \) for \( dIn \) and equation \( (29) \) for \( dH_n \) as well as equations \( (36) \) and \( (37) \) derived for generic beliefs rather than beliefs of an economist, we find

\[
dQ_P(t) = \sum_{n=0}^{N} [a_n \cdot H_n(t) + b_n \cdot Z_n(t)] \cdot dt + dB_Q^*(t), \tag{62}
\]

where \( a_0, a_n \) and \( b_n \) are defined in \( (104), (105), \) and \( (106) \) in the Appendix and \( dB_Q^* \) is defined as

\[
dB_Q^* := \frac{\sigma_G \cdot \bar{\Omega} \cdot C_G}{(r + \alpha_D)(r + \alpha_G)} \left( \tau_0^{1/2} dB_0^* + \sum_{n=1}^{N} \tau_n^{1/2} dB_n^* \right) + \frac{\sigma_D}{(r + \alpha_D)} \cdot dB_0^*, \tag{63}
\]
where $dB_n^* = \tau_n^{1/2}/\sigma_G \cdot (G^* - G) \cdot dt + dB_n$, $n = 0, \ldots, N$.

The price dynamics has a complicated form. Under any set of beliefs, prices do not follow a random walk, except at some point of time by incident. The drift term is a linear function of current signals $H_n(t)$, $n = 0, \ldots, N$ and variables $Z_n(t)$, $n = 0, \ldots, N$, which summarize the history of signals. The implied term structure of returns depends in a complicated manner on parameters of the model and provides a tool for thinking about horizons over which various anomalies take place.

The concept of market efficiency is often associated with the concept of excessive volatility. Shiller (1981) claims that stock price volatility systematically exceeds fundamental volatility and therefore markets are inefficient and irrational.

Our model illustrates that the connection between market efficiency and excessive volatility is subtle. From the perspective of an economist, the instantaneous volatility of fundamentals $\sigma_F^2$ implied by equation (61) and instantaneous volatility of prices $\sigma_P^2$ implied by equation (62) are equal to,

$$\sigma_F^2 = \left( \frac{\sigma_D}{r + \alpha_D} + \frac{\sigma_G \cdot \Omega_E \cdot \tau_0^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \right)^2 + N \cdot \frac{(\sigma_G \cdot \Omega_E)^2 \cdot \tau_e}{(r + \alpha_D)^2(r + \alpha_G)^2}.$$  \hspace{1cm} (64)

$$\sigma_P^2 = \left( \frac{\sigma_D}{r + \alpha_D} + \frac{\sigma_G \cdot \bar{\Omega} \cdot C_G \cdot \tau_0^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \right)^2 + N \cdot \frac{(\sigma_G \cdot \bar{\Omega} \cdot C_G)^2 \cdot \tilde{\tau}}{(r + \alpha_D)^2(r + \alpha_G)^2}.$$ \hspace{1cm} (65)

The volatility of fundamentals $\sigma_F^2$ depends on what an economist thinks about precision of signals. The volatility of prices $\sigma_P^2$ depends only on what traders think about precision of signals, but not on an economist’s views.

For a general set of parameters, the relation between these two volatilities is complicated. Sometimes price volatility is smaller than fundamental volatility. For example, it can be shown that if an economist assigns precision $\tau_e = \tilde{\tau}$ to each private signal, then he thinks the total precision of signals $\tau^E$ is lower than traders’ estimate $\tilde{\tau}$. This implies $\Omega^E < \bar{\Omega}$ and therefore $\sigma_P^2 < \sigma_F^2$, taking into account the dampening effect $C_G < 1$. In contrast, sometimes fundamental volatility is smaller than price volatility, as in the case when $\tau_e$ is significantly lower than $\tilde{\tau}$. Instead of looking at instantaneous volatilities, an economist usually examines returns volatility using prices sampled at specific frequen-
cies. In our model, the relation between price volatility and fundamental volatility over a particular horizon is even more complicated, because of a non-zero drift term in equation (62). Defining market efficiency as the concept related to excessive volatility is not straightforward.

This discussion suggests that a more precise definition of market efficiency is warranted. Our model implies several alternative measures of market efficiency. The first measure is the inverse of error variance $1/\Omega^E$, which quantifies the inverse of how much time the market price is behind fundamentals. The second measure is the price resiliency $\alpha_G + \Omega^E \tau^E$, which is equal to $(\alpha_G^2 + \tau^E)^{1/2}$ from equation (26); high resiliency is expected in markets with high mean-reversion of fundamentals $\alpha_G$ and high total precision carried in information flow $\tau^E$.

**D. Comparative Statics.**

Even though the six constants $\gamma_p > 0, \psi_{SS} > 0, \psi_{Sn} < 0, \psi_{nn}, \psi_{xx},$ and $\psi_{nx}$ have to be determined numerically from the system of six polynomial equations (90)-(95), we can derive several properties of equilibrium in closed form using the dimensional analysis.

Dimensional analysis is one of the powerful methods often used in physics. It says that any valid equation must apply not only to numerical quantities on both sides of equation but also to their units, since these quantities must be of the same nature. In other words, any equation must be dimensionally homogeneous. It is worth thinking about units in the three main equations describing our model: equation (22) for the evolution of dividends, equation (23) for the evolution of a growth rate, and equations (25) for the evolution of information flow.

There are three major units in our model: currency units (e.g., dollars), share units (e.g., shares), and time units (e.g., years). Dividends $D(t)$ are in dollar-per-share units ($$/s$). Growth rate $G^*(t)$ is in dollar-per-share-per-time units ($$/s/T$). Information $I_n(t)$ is in squared-root-of-time units ($T^{1/2}$). To make all equations to be consistent in terms of units, we should assign different units to parameters in the model. Parameters $r, \rho, \alpha_D, \alpha_G$ are in units per time ($1/T$). Parameter $\sigma_D$ is in dollar-per-share-per-squared-root-of-time units.
units (\$/s/T^{1/2}). Parameter $\sigma_G$ is in dollar-per-share-per-squared-root-of-time-per-time units (\$/s/T^{3/2}). Parameters $\tau_H$, $\tau_L$, and $\tau_0$ are in units per time squared (1/T^2). Also, risk aversion $A$ is in units per dollar. Since $\Omega$ from equation (26) is similar to the variance of a Sharpe ratio, it has units of time.

Any model must be invariant with respect to changes in units. For example, we could describe our model not in terms of dollars, shares, and years, but in terms of pennies, dozens of shares, and hours. Changes in units must not change main equations.

Let $(\gamma_P^*, \psi_{SS}^*, \psi_{Sn}^*, \psi_{nn}^*, \psi_{nx}^*, \psi_{xx}^*)$ be a solution to the six polynomial equations (90)-(95) for parameters $A$, $\sigma_D$, $\sigma_G$, $r$, $\alpha_G$, $\alpha_D$, $\tau_0$, $\tau_L$, and $\tau_H$. Suppose currency units change by $K$, share units change by $B$, and time units change by $T$. Based on the dimensional analysis, $(\gamma_P^* \cdot \frac{T^2}{K \cdot B}, \psi_{SS}^* \cdot B^2, \psi_{Sn}^* \cdot B \cdot T^{1/2}, \psi_{nn}^* \cdot T, \psi_{nx}^* \cdot \frac{T}{B}, \psi_{xx}^* \cdot T)$ is a solution for the system with parameters $A/K, \sigma_D \cdot K \cdot B \cdot T^{1/2}, \sigma_G \cdot K \cdot B \cdot T^{3/2}, r \cdot T, \alpha_G \cdot T, \alpha_D \cdot T, \tau_0 \cdot T^2, \tau_L \cdot T^2$, and $\tau_H \cdot T^2$. Consequently, $C_H$ changes to $C_H \cdot \frac{T^{3/2}}{B}$, $C_L$ changes to $C_L \cdot \frac{T^{1/2}}{B}$, $\gamma_S$ changes to $\gamma_S \cdot T$, and $C_G$ does not change.

This intuition helps to derive some comparative statics properties of the solution, for example, how it depends on traders’ risk aversion. Scaling currency units by $K = 1/F$ and share units by $B = F$ changes risk aversion from $A$ to $A \cdot F$, while other parameters remain unchanged. Consequently, the coefficient $C_H$ changes to $C_H / F$, and the coefficient $C_L$ changes to $C_L / F$. i.e., when risk aversion increases ($F > 1$), traders reduce their target inventories and trade less aggressively. The coefficients $\gamma_S$ and $C_G$ do not change. The changes in risk aversion leads to changes in quantities, but prices and the speed of trading games remain the same. In some sense, risk aversion coefficient is not a deep parameter of the model, but the parameter fixing the units in which wealth is measured and utility functions are defined.

The dimensional analysis helps to identify which parameters in the model are important and which parameters can be re-scaled to be equal to one without loss of generality. By re-scaling parameters of the model we can effectively reduce the number of exogenous variables from ten to seven. We can assume that $r = 0.01$ (time units), $\sigma_G = 1$ (share units), and $A = 1$ (dollar units). For our benchmark case, we assume the following parameter values: $r = 0.01, A = 1, \alpha_D = 0.01, \alpha_G = 0.1, \sigma_G = 1, \sigma_D = 100, \tau_0 = \frac{\sigma_G^2}{\sigma_D} =$
Figure 2: $C_H$ and $C_L$ against $N$ while fixing $\bar{\tau}$. The parameter values are $r = 0.01, A = 1, \alpha_D = 0.01, \alpha_G = 0.1, \sigma_G = 1, \sigma_D = 100, \tau_0 = \frac{\sigma_D}{\sigma_G} = 0.0001, \tau_H = 0.4, \bar{\tau} = 2$.

To analyze the effect of increasing competition in the market, we plot coefficients $C_L$, $C_H$, $C_G$, and $\gamma_S$ as functions of the number of traders $N$ while keeping the total precision $\bar{\tau}$ fixed. There is the same amount of information but information is dispersed among more market participants. Note that when $N$ increases the effective risk aversion of the market $A/N$ changes as well. The market converges to a risk-neutral case.

Figure 2 shows how coefficients $C_H$ and $C_L$ change against $N$ while fixing $\bar{\tau}$. The coefficient $C_L$ defining target inventories increases with $N$ and converges to the constant level after $N = 150$. Each trader thinks that when $N$ is relatively large, the other traders in the market become almost risk-neutral in aggregate. Their inventory positions are therefore similar to the positions they would hold in the competitive market with many risk-neutral counterparties. The constant $C_H$ also increases with $N$ but does not converge to any constant level. As $N$ increases, the traders trade more aggressively since the risk bearing capacity of the market becomes infinitely large.

Figure 3 shows how coefficients $\gamma_S$ and $C_G$ change against $N$ while fixing $\bar{\tau}$. The speed of adjusting positions denoted by $\gamma_S$ increases with the number of traders $N$, as it becomes less and less costly for traders to trade aggressively towards their target inventories. The
coefficient $C_G$ is monotonically decreasing with $N$. This coefficient is always less than one, being the closest to one when it is equal to $C_G = 0.998$ for $N = 19$. This fact is important for our intuition about monopolistic competition with disagreement. Traders take into account that others are too overconfident and trade too much in response to their signals. This thinking is internalized in equilibrium prices which become less sensitive to the average signal than in the model with no disagreement.

We also analyze the effect of disagreement by looking at coefficients $C_L$, $C_H$, $C_G$, and $\gamma_S$ as functions of the precision $\tau_H$ of traders’ own private signal. As before, we keep the total precision $\bar{\tau}$ fixed. Note that the equilibrium exists only when $\tau_H$ is sufficiently large relative to $\tau_L$. For example, in one-period model, the second order condition holds only when $(N - 2)\tau_H - 2(N - 1)\tau_L > 0$, i.e., $\tau_H$ is more than twice bigger than $\tau_L$. In continuous time model, the traders also have to be sufficiently overconfident, otherwise the numeric algorithm for solving the system (90)-(95) does not converge to the solution.

Figure 3 and figure 5 shows coefficients $C_L$, $C_H$, $C_G$, and $\gamma_S$ as the function of $\tau_H$ keeping total precision $\bar{\tau}$ fixed. The coefficient $C_G$ starts from values close to one, for example, $C_G = 0.998$ when $\tau_H = 0.08$ and $\tau_L = 0.02$, and then decreases monotonically as $\tau_H$ increases. The more traders disagree with each other, the more traders discount actions.
of other traders and therefore dampen the equilibrium price changes in response to the average signal. The coefficients $C_H$ and $C_L$ exhibit very similar patterns, as $\tau_H$ changes. Both coefficients increase, as the degree of disagreement increases. The coefficient $C_L$ related to target inventories is a non-monotonic function of $\tau_H$. It first increases with $\tau_H$, since traders become more overconfident in their own signal, but then decreases.

IV. Liquidity and Transaction Costs

Our paper generates insights on how transaction costs depend on the speed of trading and what price dynamics is triggered in response to execution of large orders.

A. Price Impact Functions.

Suppose an $N+1$ out-of-equilibrium trader silently enters the market as a liquidity trader and trades starting from his zero inventory towards a target inventory $\bar{S}$. He buys if $\bar{S} > 0$ and sells if $\bar{S} < 0$. His trading strategy is defined by profile of his inventory $S(t)$ and implies a trading rate $x(t)$ such that $dS(t) = x(t) \cdot dt$. His trades create price pressure,
Figure 5: $\gamma_S$ and $C_G$ against $\tau_H$ while fixing $\tilde{\tau}$. The parameter values are $r = 0.01$, $A = 1$, $\alpha_D = 0.01$, $\alpha_G = 0.1$, $\sigma_G = 1$, $\sigma_D = 100$, $\tau_0 = \frac{\sigma^2}{\sigma_D} = 0.0001$, $N = 100$, $\tilde{\tau} = 2$.

and the market price deviates from the equilibrium price path $P^*$ by a margin

$$P(S(t), x(t)) - P^* = \lambda_0 + \lambda_S \cdot S(t) + \lambda_x \cdot x(t),$$  \hspace{1cm} (66)

where constants $\lambda_0$, $\lambda_S$ and $\lambda_x$ are define as,

$$\lambda_0 := 0, \quad \lambda_S := \frac{1}{(N-1) \cdot \gamma_p}, \quad \lambda_x := \frac{1}{(N-1) \cdot \gamma_p}. \hspace{1cm} (67)$$

Similarly, if trader $n$ in the model implemented a trading strategy $x(t)$ instead of his optimal strategy $x^*(t)$, then his inventory trajectory $S(t)$ would differ from the optimal inventory trajectory $S^*(t)$ and the market prices would satisfy equation (66) with $\lambda_0 := -\lambda_S \cdot S^*(t) - \lambda_x \cdot x^*(t)$.

The first term $\lambda_S \cdot S$ in price impact function (66) represents the permanent component, linear in the number of shares traded. The second term $\lambda_x \cdot x$ is the temporary component determined by the derivative of inventory $x$, i.e., the speed of trading: Speeding up trading increases its magnitude and slowing down trading reduces it. The temporary component does not exist in equilibrium price impact functions in Kyle (1985), where the cost of continuous trading does not depend on the speed of trading, $\lambda_x = 0$.

Similar price impact models have been used by financial professionals who have recog-
nized long time ago that the trading speed affects realized costs. For example, the same model can be found in Grinold and Kahn (1995) as well as Almgren and Chriss (2000), who considered optimal trading algorithms for execution of a given order in the presence of exogenously specified price impact functions. Obizhaeva and Wang (2013) suggested an alternative model with the temporary price impact decaying gradually over time at an exponential rate rather than instantaneously. These frameworks and their generalizations have been proven to be remarkably useful in constructing trading systems and analyzing transaction costs.

The price impact function (66) allows us to examine how the speed of execution interacts with intertemporal properties of liquidity and affects transaction costs. We consider three cases.

First, suppose a trader has to buy or sell $\tilde{S}$ shares of a risky asset during the period $[0, T]$. If he trades at a constant rate $x$, then price $P(t) = P^* + \lambda_S \cdot x \cdot t + \lambda_x \cdot x$. In our model, prices do not follow a martingale and a trader may have a particular view on how prices will be changing in the future. We therefore modify the concept of implementation shortfall of Perold (1988) and examine the transaction cost relative to the price dynamics “unaffected” by execution,

\[ E\{C\} = E\left\{ \int_{t=0}^{T} (P(t) - P^*(t)) \cdot x \cdot dt \right\} = \left( \lambda_S + \frac{\lambda_x}{T/2} \right) \cdot \tilde{S}^2. \]  

(68)

If horizon $T$ goes to infinity, then the temporary impact costs can be totally avoided and the transaction costs converge to permanent impact costs, $\lambda_S \cdot \frac{\tilde{S}^2}{2}$.

Second, suppose a trader wants to minimize not only the expected costs but also its variance, $E\{C\} + \frac{R}{2} \cdot Var\{C\}$, as considered by Grinold and Kahn (1999). If the execution horizon $T$ is finite, they show that optimal strategy is described by $x(t) = k \cdot \tilde{S} \cdot \frac{\cosh(k(T-t))}{\sinh(kT)}$ and $S(t) = \tilde{S} - \tilde{S} \frac{\sinh(k(T-t))}{\sinh(kT)}$, where $k := \left( \frac{R \cdot \sigma^2}{2 \lambda_x} \right)^{1/2}$. When execution is unconstrained by time, the optimal strategy converges to $x(t) = k \cdot (\tilde{S} - S(t))$, $S(t) = \tilde{S} \cdot (1 - e^{-k \cdot t})$, i.e. monotonic convergence to target level at a rate $k$. The parameter $1/k$ is an intrinsic “half-life” of execution in unconstrained case. Trades are implemented faster when risk aversion $R$ and price volatility $\sigma$ are large relative to the temporary price impact $\lambda_x$. The
total costs of buying or selling $S$ shares optimally in absence of any time constraint is equal to $(\lambda_S/2 + \lambda_x \cdot k) \cdot \bar{S}^2$. When execution horizon is short relative to intrinsic execution time, these costs are amplified by inflated temporary price impact.

Third, suppose a trader trades at a rate $x(t) = \gamma \cdot (\bar{S} - S)$. His inventory then evolves as $S(t) = \bar{S} \cdot (1 - e^{-\gamma \cdot t})$, and he incurs the costs $E\{C\} = E\{\int_0^\infty (\lambda_S \cdot x + \lambda_x \cdot \gamma (\bar{S} - S)) \cdot x \cdot dt\}$. Since $\lambda_S = \gamma_S \cdot \lambda_x$ from equation (67),

$$E\{C\} = \left(\lambda_S + \gamma \cdot \lambda_x\right) \cdot \frac{\bar{S}^2}{2} = \lambda_S \cdot \left(1 + \frac{\gamma}{\gamma_S}\right) \cdot \frac{\bar{S}^2}{2}.$$  \hspace{1cm} (69)

All three examples reveal similar properties. The fast execution algorithm leads to the large temporary price impact. Block trades with $\gamma = \infty$ are infinitely expensive. As $\gamma \to 0$, a trader walks up the demand schedule as a price discriminating monopolist and incurs the costs $E\{C\} = \lambda_S \cdot \frac{\bar{S}^2}{2}$, because the temporary costs are converging to zero.

In the equilibrium, the speed of execution $\gamma$ is equal to $\gamma_S$. Since everybody is trying to walk up the demand schedule of others by slowing down his trading, traders can not trade as price discriminating monopolists. Price adjust instantaneously. Their total costs are equal to $E\{C\} = \lambda_S \cdot \bar{S}^2$. The costs due to permanent price impact are exactly equal to costs due to temporary price impact. In other words, each trader pays out his potential monopolist’s profits to others in a form of temporary price impact. In symmetric equilibrium, everybody expects to break even, because it is a common knowledge that, on average, trades do not have any information.

Our model links the price impact coefficients $\lambda_S$ and $\lambda_x$ to particular features of the market. Figure 6 shows that both coefficients decrease as the number of traders $N$ increases, keeping the total precision fixed. The market becomes less risk-averse in aggregate ($A/N$ decreases) and traders are willing to provide more liquidity to each other, because they believe that others have less precise information ($\tau_L$ decreases). The speed at which traders adjust their positions towards target levels increases. Figure 7 shows that both coefficient also decrease in the degree of disagreement ($\tau_H$ increases), since traders are willing to provide more liquidity to presumably less informed counterparties. Also, the dimensional analysis implies that if risk aversion $A$ changes to $A/F$, then price impact...
Figure 6: $\frac{1}{\lambda_S}$ and $\frac{1}{\lambda_x}$ against $N$ while fixing $\bar{\tau}$. The parameter values are $r = 0.01, A = 1, \alpha_D = 0.01, \alpha_G = 0.1, \sigma_G = 1, \sigma_D = 100, \tau_0 = \frac{\sigma_G^2}{\sigma_D}, \tau_H = 0.4, \bar{\tau} = 2$.

Figure 7: $\frac{1}{\lambda_S}$ and $\frac{1}{\lambda_x}$ against $\tau_H$ while fixing $\bar{\tau}$. The parameter values are $r = 0.01, A = 1, \alpha_D = 0.01, \alpha_G = 0.1, \sigma_G = 1, \sigma_D = 100, \tau_0 = \frac{\sigma_G^2}{\sigma_D}, \bar{\tau} = 2$. 

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coefficients $\lambda_S$ and $\lambda_x$ change to $\lambda_S/F$ and $\lambda_x/F$.

**B. Flash Crashes and Speed of Trading.**

From the perspective of each oligopolistic trader, trading strategy is “partial adjustment” towards “steady-state” inventory, but a trader may choose the speed of adjustment $\gamma$ differently from the optimal speed of adjustment $\gamma_s$. What happens with price dynamics if a trader deviates from the equilibrium and trades faster or slower than his equilibrium rate of trading?

Suppose trader $n$ calculates his path of a target inventory as in the equilibrium, $S_{TI}^n = C_L \cdot (H_n - H_{-n})$, but he trades at a rate $\gamma$ instead of $\gamma_s$ starting at time $t$,

$$x_n(t + T) = \gamma \cdot (S_{TI}^n(t + T) - S_n(t + T)). \quad (70)$$

The level of his inventory has the following trajectory

$$S_n(t + T) = e^{-\gamma T} \cdot \left( S_n(t) + \int_{t=T}^{t=T} e^{-\gamma (t-u)} \cdot \gamma \cdot C_L \cdot (H_n(u) - H_{-n}(u)) \cdot du \right). \quad (71)$$

From the perspective of a trader, the price is linear function of his inventory (permanent price impact) and the derivative of his inventory (temporary impact), as shown in equation (45). When a trader deviates from his equilibrium strategy, the price $P(t + T)$ deviates from its equilibrium path $P^*(t + T), T \geq 0$,

$$P(t+T)-P^*(t+T) = \frac{\psi_{SS}}{\gamma A} \cdot (S_n(t+T) - S_n^*(t+T)) + \frac{1}{(N-1)\gamma P} \cdot (x_n(t+T) - x_n^*(t+T)), \quad (72)$$

where constants $\psi_{SS}$ and $\gamma_P$ are determined by the system of equations (90)-(95) in the Appendix.

Taking into account that $S_n(t) = S_n^*(t)$ at time $t$ and plugging equations for $x_n^*(t + T)$, $x_n(t + T)$, $S_n^*(t + T)$, and $S_n(t + T)$ into equation (72) yield the deviation of prices from
their equilibrium path,

\[ P(t+T) - P^*(t+T) = S_n(t) \cdot \frac{\gamma S - \gamma}{(N-1)\gamma P} \cdot e^{-\gamma T} + C_L \frac{\gamma_S - \gamma}{(N-1)\gamma P} \cdot (H_n(t+T) - H_n(t+T)) + \]

\[ + \int_{u=t}^{t+T} C_L \cdot (H_n(u) - H_n(u)) \cdot \frac{\gamma}{(N-1)\gamma P} \cdot e^{-\gamma (t+T-u)} \cdot \] \( S_{n}(t) \cdot e^{-T} - (N-1)P \cdot e^{-T} + C_L - S_{n}(t) \cdot e^{-T} - (N-1)P \cdot e^{-T} \cdot \frac{\gamma S - \gamma}{(N-1)\gamma P} \cdot du. \]

(73)

Using recursive formula (100) for \( H_n(u) - H_n(u) \) derived in Appendix, we can write the expected price deviation \( \mathbb{E}\{P(t+T) - P^*(t+T)|\Phi_t\} \) given information at time \( t \) as,

\[ \mathbb{E}\{P(t+T) - P^*(t+T)|\Phi_t\} = \]

\[ = \frac{\gamma - \gamma S}{(N-1)\gamma P} \left[ -e^{-\gamma T} \cdot S_n(t) + \frac{e^{-\gamma T} - (\alpha_G + \bar{\Omega})e^{-(\alpha_G + \bar{\Omega})T}}{\gamma - \alpha_G - \bar{\Omega}} \cdot S_{n}^{TI}(t) + \right] \]

\[ + G(t) \frac{C_L(\tau_n^{1/2} - \frac{1}{N-1} \sum_{i=1,i\neq n}^{N} \tau_i^{1/2})}{\tau_G \cdot \bar{\Omega} \cdot \bar{T}} \left[ \frac{(\alpha_G + \bar{\Omega})e^{-(\alpha_G + \bar{\Omega})T} - \gamma \cdot e^{-\gamma T} + \gamma \cdot e^{-\gamma T} - \alpha_G \cdot e^{-\alpha_G T}}{\gamma - \alpha_G} \right]. \]

(74)

If a trader follows his optimal strategy and \( \gamma = \gamma S \), the prices change along the equilibrium path and \( \mathbb{E}\{P(t+T) - P^*(t+T)|\Phi_t\} = 0 \). For symmetric beliefs, the last term is equal to zero. As \( T \to \infty \), \( \mathbb{E}\{P(t+T) - P^*(t+T)|\Phi_t\} \to 0 \), and the price gap disappears.

Figure 8 illustrates how the speed of trading affects price dynamics. A trader starts with no inventory \( S_n(t) = 0 \) and sells shares to reach target inventory \( S_{n}^{TI}(t) = -1 \). As a benchmark, the solution for the parameters \( r = 0.01, A = 1, \alpha_D = 0.01, \alpha_G = 0.1, \sigma_D = 100, \tau_0 = 0.0001, \tau_L = 0.009, \tau_H = 0.1, \bar{T} = 1.0001, N = 100 \) implies the optimal speed of trading \( \gamma S_1 = 0.6352 \). Figure 8 depicts the difference between off-equilibrium price path and equilibrium price path, calculated from equation (74), when a trader speeds up or slows down his trading.

When a trader sells at a rate twice slower than equilibrium rate \( (\gamma_1 = \gamma S_1/2) \), the price does not decline as much as along the equilibrium path; this initial positive price gap slowly disappears as the time passes. Slow execution reduces transitory price impact, but the price eventually converges to equilibrium level. When a trader sells at a rate twice faster than equilibrium rate \( (\gamma_2 = 2 \cdot \gamma S_1) \), the price immediately drops relative to the

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Deviations of Price Paths from Equilibrium Levels

Figure 8: $E[P(t + T) - P^*(t + T)]_t$ against time period $T$ for $\gamma_1 = \gamma_{S1}/2$ (red curve) and $\gamma_2 = 2\gamma_{S1}$ (green curve). The parameter values are $r = 0.01, A = 1, \alpha_D = 0.01, \alpha_G = 0.1, \sigma_G = 1, \sigma_D = 100, \tau_0 = \frac{\sigma_G^2}{\sigma_D^2} = 0.0001, N = 100, \tau_L = 0.009, \tau_H = 0.1, S_n(t) = 0, S_{nT}(t) = -1$. And $E[P(t+T)-P^*(t+T)]_t$ against time period $T$ for $\gamma_3 = \gamma_{S2}/2$ (purple curve) and $\gamma_4 = 2\gamma_{S2}$ (orange curve) when $r_0, \tau_L$ and $\tau_H$ are doubled.

equilibrium path; the initial negative price gap reverts to equilibrium path over time and price response exhibits a distinct V-shaped pattern. Both profiles are depicted in solid lines.

The price responses depends on the total precision in the economy. For example, if the total precision increases from $\bar{\tau} = 1.0001$ to $\bar{\tau} = 2.0002$ then slowing down or speeding up trading relative to the optimal partial adjustment speed of $\gamma_{S2} = 0.8935$ leads to smaller initial price gaps and faster convergence to the equilibrium levels, as depicted in dotted lines on figure 8.

There is a distinct asymmetry in price responses: Faster execution exerts a bigger price pressure on the market prices than slower execution. For example, when the execution speed doubles, the magnitude of initial negative price deviation is twice bigger than the magnitude of initial positive price deviation when the execution speed halves. Indeed,
\[ E\{P(t + T) - P^*(t + T)|\Phi_t\} \to -\frac{\gamma}{(N-1)\gamma_p} \cdot (S_{T I}^n(t) - S_n(t)) \text{ as } T \to 0 \text{ in equation (74).} \]

Plugging in \( \gamma_1 = \gamma_{S1}/2 \) and \( \gamma_2 = 2 \cdot \gamma_{S1} \) derives this asymmetry quantitatively.

Given parameters of the model, this analysis provides a framework for calibrating the V-shaped price patterns observed during flash crashes and probably related to very fast execution of large orders, as suggested by Kyle and Obizhaeva (2013). Flash crashes are associated with significant temporary price changes, excessive volatility, and V-shaped price recoveries. One of the famous examples is the Flash crash on May 6, 2010, where the E-mini S&P 500 futures price plunged by 5.12% over a five-minute period. Pre-programmed circuit breakers triggered a five-second pause in trading, after which the market quickly recovered all of the earlier losses. In the aftermath of the flash crash, Staffs of the CFTC and SEC (2010a,b) issued a joint report indicating that the flash crash was triggered by an automated execution algorithm that sold 75,000 S&P 500 E-mini futures contracts worth approximately $4 billion. A detailed analysis can be found in Kirilenko et al. (2012). Market microstructure invariance would imply the price impact of only 0.61% rather than actual drop in price of 5.12%. Kyle and Obizhaeva (2013) attribute the difference to unusually fast execution of that order. Indeed, the order was executed over a twenty-minute period between 2:32 p.m. and 2:51 p.m. ET, while orders of similar magnitude have been usually executed over a much longer period of time, often stretching over several days.

V. Conclusion

We develop a model with overconfidence and market power where traders agree to disagree based on the flow of public and private information and trade with each other, taking into account their market power. This model provides a framework for thinking about intertemporal properties of market liquidity, transaction costs, and market prices. We take the idea that traders smooth order flow over time to its logical limit and formally prove the intuition of Black (1995).

The idea that securities markets offer a flow equilibrium rather than a stock equilibrium may seem far-fetched at first glance. We believe, however, that in the future, it is quite
possible that market will offer explicit flow equilibrium trading mechanisms, with market clearing taking place in terms of flows rather than stocks. There are already several ways in which the current institutional features of trading illustrate the importance of these concepts.

First, the execution is often automated, and many orders are shredded into sequences of one-lot trades executed at very high frequencies, sometimes within nanoseconds. Traders may explicitly set the speed with which their computers are firing their trades into the market place.

Second, some existing contracts are essentially equivalent to continuous smooth trading strategies implemented indirectly via indexing rather than directly via explicit flow demand schedules, as for example, VWAP or TWAP contracts indexed to volume weighted prices or time weighted prices over a particular interval of time. They represent traders’ attempts to trade flows in the current institutional structure. Interest rate swaps represent similar strategies over longer horizons (e.g., three months or six months rather than minute-by-minute or day-by-day).

Third, trades in which customers participate alongside the specialist also represent a type of continuous flow trading strategy. Fourth, certain exotic options can be viewed as automatic updates of continuous flow equilibrium trading strategies (for example, a call option on the average price of an asset over an interval of time).

The continuous flow equilibrium model represents an abandonment of the arbitrage argument that underlies the Black-Scholes formula. Perfect delta-hedging requires an infinitely impatient trading strategy, which is infinitely expensive in a continuous flow equilibrium. Thus, we need an equilibrium argument to explain option prices, even in markets with constant volatility. In the future, options market would probably play the role of pricing infinitely impatient trading strategies.
References


Appendix

**Proof of Theorem 1:** For the second order condition to be positive, we need to have
\[
\frac{2}{(N-1)^2} + \frac{A}{\tau} > 0, \text{ i.e., }
\]
\[
\frac{A}{\tau} \cdot \left( \frac{N \tau_H}{N-1} \right) > 0.
\]
Therefore, the second order condition holds if and only if \((N-2)\tau_H - 2(N-1)\tau_L > 0\).
Substituting (13) and (14) into (10), we get trader \(n\)’s optimal demand \(x_n^*\). Substituting it into the market clearing condition \(\sum_{m=1}^{N} X_m(i_0, i_m, p) = 0\), we get the equilibrium price \(P^*\).

Q.E.D.

**Proof of Theorem 2:** Trader \(n\) thinks that public and private signals are changing as follows:
\[
dH_0(t) = -(\alpha_G + \Omega \cdot \tau) \cdot H_0(t) \cdot dt + \tau_0^{1/2} \cdot G_n(t) \cdot dt + dB_0^*.
\]
\[
dH_n(t) = -(\alpha_G + \Omega \cdot \tau) \cdot H_n(t) \cdot dt + \tau_H^{1/2} \cdot G_n(t) \cdot dt + dB_n^*,
\]
\[
dH_{-n}(t) = -(\alpha_G + \Omega \cdot \tau) \cdot H_{-n}(t) \cdot dt + \tau_L^{1/2} \cdot G_n(t) \cdot dt + \frac{1}{N-1} \sum_{m=1, m \neq n}^{N} dB_m^*,
\]
where \(G_n(t)\) is trader \(n\)’s estimate of growth rate at time \(t\) defined in equation (34), \(dB_D^* = \tau_0^{1/2}(G^*(t) - G_n(t))dt + dB_D, dB_n^* = \tau_H^{1/2}(G^*(t) - G_n(t))dt + dB_n,\) and \(dB_m^* = \tau_L^{1/2}(G^*(t) - G_n(t))dt + dB_m\).

Let \(V(M, S, D, \hat{H}_n, \hat{H}_{-n})\) be the value function at time \(t\) for the optimal consumption and investment policy \((c, x)\),
\[
V(M, S, D, \hat{H}_n, \hat{H}_{-n}) := \max_{\{c_t, x_t\}} \mathbb{E}_t \left[ \int_{s=t}^{\infty} -e^{-\rho(s-t)} \cdot e^{-Ac_s} \cdot ds \right],
\]
where state variables satisfy stochastic differential equations
\[
dM(t) = (r \cdot M(t) + S(t) \cdot D(t) - c_t - P(x_t) \cdot x_t) \cdot dt
\]
\[ dS(t) = x_t \cdot dt, \quad (81) \]
\[ dD(t) = -\alpha_D \cdot D(t) \cdot dt + G_n(t) \cdot dt + \sigma_D \cdot dB_D + (G^*(t) - G_n(t)) \cdot dt. \quad (82) \]
\[ d\hat{H}_n(t) = -(\alpha_G + \hat{\Omega}) \cdot \hat{H}_n(t) \cdot dt + (\tau_H^{1/2} + \hat{A} \tau_0^{1/2}) \cdot (\hat{\Omega} \tau_H^{1/2} \cdot \hat{H}_n + \hat{\Omega}(N-1) \tau_L^{1/2} \cdot \hat{H}_n) \cdot dt + d\hat{B}_n + \hat{A} \hat{d}B_D^*, \quad (83) \]
\[ d\hat{H}_{-n}(t) = -(\alpha_G + \hat{\Omega}) \cdot \hat{H}_{-n}(t) \cdot dt + (\tau_L^{1/2} + \hat{A} \tau_0^{1/2}) (\hat{\Omega} \tau_L^{1/2} \cdot \hat{H}_n + \hat{\Omega}(N-1) \tau_L^{1/2} \cdot \hat{H}_{-n}) dt \]
\[ + \frac{1}{N-1} \sum_{m=1, m \neq n}^N d\hat{B}_m + \hat{A} \hat{d}B_D^*, \quad (84) \]

where \( d\hat{B}_D^* = \tau_0^{1/2} \sigma_G^{-1}(G^*(t) - G_n(t)) dt + dB_D \), \( d\hat{B}_n^* = \tau_H^{1/2} \sigma_G^{-1}(G^*(t) - G_n(t)) dt + dB_n \), and \( d\hat{B}_m^* = \tau_L^{1/2} \sigma_G^{-1}(G^*(t) - G_n(t)) dt + dB_m \).

Then the HJB equation is
\[
\max_{c,x} \left\{ u(c) - \rho V + \frac{\partial V}{\partial M} (rM + SD - c - Px) + \frac{\partial V}{\partial S} \right\} + \frac{\partial V}{\partial D} (-\alpha_D + \sigma_G \hat{\Omega} \sqrt{\tau_H} \hat{H}_n + \sigma_G \hat{\Omega}(N-1) \sqrt{\tau_L} \hat{H}_{-n}) \\
+ \frac{\partial V}{\partial \hat{H}_n} \left( -(\alpha_G + \hat{\Omega}) \hat{H}_n(t) + (\sqrt{\tau_H} + \hat{A} \sqrt{\tau_0}) (\hat{\Omega} \sqrt{\tau_H} \hat{H}_n + \hat{\Omega}(N-1) \sqrt{\tau_L} \hat{H}_{-n}) \right) \\
+ \frac{\partial V}{\partial \hat{H}_{-n}} \left( -(\alpha_G + \hat{\Omega}) \hat{H}_{-n}(t) + (\sqrt{\tau_L} + \hat{A} \sqrt{\tau_0}) (\hat{\Omega} \sqrt{\tau_L} \hat{H}_n + \hat{\Omega}(N-1) \sqrt{\tau_L} \hat{H}_{-n}) \right) \frac{1}{2} \frac{\partial^2 V}{\partial D^2} \\
+ \frac{\partial^2 V}{\partial \hat{H}_n^2} (1 + \hat{A}^2) + \frac{\partial^2 V}{\partial \hat{H}_{-n}^2} \left( \frac{1}{N-1} + \hat{A}^2 \right) + \left( \frac{\partial^2 V}{\partial D \partial \hat{H}_n} + \frac{\partial^2 V}{\partial D \partial \hat{H}_{-n}} \right) \hat{A} \sigma_D + \frac{\partial^2 V}{\partial H_0 \partial \hat{H}_n} \hat{A}^2 = 0. \tag{85} \]

For the value function of a conjectured form (46), the Hamilton-Jacobi-Bellman (HJB) equation becomes
\[
\min_{c,x} \left\{ -e^{-Ac} - \rho + \psi_M (rM + SD - c - Px) + (\psi_{S S} + \psi_{S D} D + \psi_{S n} \hat{H}_n + \psi_{S x} \hat{H}_{-n}) x \right\} \\
+ \psi_{S D} (\alpha_D + \sigma_G \hat{\Omega} \sqrt{\tau_H} \hat{H}_n + \sigma_G \hat{\Omega}(N-1) \sqrt{\tau_L} \hat{H}_{-n}) \\
+ (\psi_{S n} + \psi_{n n} \hat{H}_n + \psi_{n x} \hat{H}_{-n}) \left( -(\alpha_G + \hat{\Omega}) \hat{H}_n(t) + (\sqrt{\tau_H} + \hat{A} \sqrt{\tau_0}) (\hat{\Omega} \sqrt{\tau_H} \hat{H}_n + \hat{\Omega}(N-1) \sqrt{\tau_L} \hat{H}_{-n}) \right) \]

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Maximizing the HJB equation over optimal consumption and investment policy \((c, x)\), we get equations (47) and (48).

Substituting (50) into equation (48), we get trader \(n\)'s optimal demand \(x_n(t)^*\) in equation (53). Substituting (50) and (53) into (45), we get the equilibrium price \(P^*\). Substituting (47) and (48) back into the HJB equation (86) and setting the constant term, coefficients of \(M, SD, S^2, S\hat{H}_n, S\hat{H}_{-n}, \hat{H}^2_n, \hat{H}^2_{-n}\) and \(\hat{H}_n\hat{H}_{-n}\) to be zero, we get

\[
\psi_M = -rA, \tag{87}
\]
\[
\psi_{SD} = -\frac{rA}{r + \alpha_D}, \tag{88}
\]
\[
\psi_0 = 1 - \log r + \frac{1}{r} \left( -\rho + \frac{1}{2} (1 + \hat{A}^2) \psi_{nn} + \frac{1}{2} \left( \frac{1}{N - 1} + \hat{A}^2 \right) \psi_{xx} + \hat{A}^2 \psi_{nx} \right). \tag{89}
\]

In addition, by setting the coefficients of \(S^2, S\hat{H}_n, S\hat{H}_{-n}, \hat{H}^2_n, \hat{H}^2_{-n}\) and \(\hat{H}_n\hat{H}_{-n}\) to be zero, we get the following six polynomial equations about six unknowns \(\gamma_P, \psi_{SS}, \psi_{Sn}, \psi_{nn}, \psi_{xx},\) and \(\psi_{nx}\).

\[
S^2 : -\frac{1}{2} r \psi_{SS} - \frac{\gamma_P (N - 1)}{rA} \psi_{SS}^2 + \frac{r^2 A^2 \sigma_D^2}{2 (r + \alpha_D)^2} + \frac{1}{2} \left( 1 + \hat{A}^2 \right) \psi_{Sn}^2 + \frac{1}{2} \left( \frac{1}{N - 1} + \hat{A}^2 \right) \frac{(N - 2)^2}{4} \psi_{Sn}^2
\]
\[
- \frac{rA}{r + \alpha_D} \hat{A} \sigma_D \frac{N}{2} \psi_{Sn} + \hat{A}^2 \frac{N - 2}{2} \psi_{Sn}^2 = 0, \tag{90}
\]
\[ S\hat{H}_n : -r\psi_n + \frac{\gamma_p(N-1)}{rA} \psi_{SS} \psi_n - \frac{rA}{r + \alpha_D} \sigma_G \Omega \sqrt{\tau_H} + a_1 \psi_n + \frac{N-2}{2} a_4 \psi_n + (1 + \hat{A}^2) \psi_{nn} \psi_n \\
+ \frac{N-2}{2} \left( \frac{1}{N-1} + \hat{A}^2 \right) \psi_{nn} \psi_n - \frac{rA}{r + \alpha_D} \hat{A} \sigma_D (\psi_{nn} + \psi_{nx}) + \hat{A}^2 \psi_{nx} \psi_n + \frac{N-2}{2} \hat{A}^2 \psi_{nn} \psi_n = 0, \] 

(91)

\[ \hat{S}\hat{H}_n : -r \frac{N-2}{2} \psi_n + \frac{\gamma_p(N-1)}{rA} \psi_{SS} \psi_n - \frac{rA}{r + \alpha_D} \sigma_G \Omega (N-1) \sqrt{\tau_L} + \left( a_3 + \frac{N-2}{2} a_2 \right) \psi_n + (1 + \hat{A}^2) \psi_{nn} \psi_{nx} \\
+ \frac{N-2}{2} \left( \frac{1}{N-1} + \hat{A}^2 \right) \psi_{xx} \psi_n - \frac{rA}{r + \alpha_D} \hat{A} \sigma_D (\psi_{xx} + \psi_{nx}) + \hat{A}^2 \psi_{xx} \psi_n + \frac{N-2}{2} \hat{A}^2 \psi_{nx} \psi_n = 0, \] 

(92)

\[ \hat{H}^2_n : - \frac{r}{2} \psi_{nn} - \frac{\gamma_p(N-1)}{4rA} \psi_{nn}^2 + a_1 \psi_{nn} + a_4 \psi_{nx} + \frac{1}{2} (1 + \hat{A}^2) \psi_{nn}^2 + \frac{1}{2} \left( \frac{1}{N-1} + \hat{A}^2 \right) \psi_{nx}^2 + \hat{A}^2 \psi_{nn} \psi_{nx} = 0, \] 

(93)

\[ \hat{H}^2_{-n} : - \frac{r}{2} \psi_{xx} - \frac{\gamma_p(N-1)}{4rA} \psi_{xx}^2 + a_2 \psi_{xx} + a_3 \psi_{nx} + \frac{1}{2} (1 + \hat{A}^2) \psi_{xx}^2 + \frac{1}{2} \left( \frac{1}{N-1} + \hat{A}^2 \right) \psi_{nx}^2 + \hat{A}^2 \psi_{xx} \psi_{nx} = 0, \] 

(94)

\[ \hat{H}_n \hat{H}_{-n} : - r \psi_{nx} + \frac{\gamma_p(N-1)}{2rA} \psi_{nn}^2 + a_3 \psi_{nn} + a_4 \psi_{xx} + (a_1 + a_2) \psi_{nx} + (1 + \hat{A}^2) \psi_{nn} \psi_{nx} \\
+ \left( \frac{1}{N-1} + \hat{A}^2 \right) \psi_{xx} \psi_{nx} + \hat{A}^2 (\psi_{nn} \psi_{xx} + \psi_{nx}^2) = 0, \] 

(95)

where

\[ a_1 = -\alpha_G - \hat{\Omega} \bar{\tau} + \hat{\Omega} \sqrt{\tau_H} (\sqrt{\tau_H} + \hat{A} \sqrt{\tau_0}), \quad a_2 = -\alpha_G - \hat{\Omega} \bar{\tau} + \hat{\Omega} (N-1) \sqrt{\tau_L} (\sqrt{\tau_L} + \hat{A} \sqrt{\tau_0}), \]
\[
\begin{align*}
\quad a_3 &= (\sqrt{\tau_L} + \hat{A}\sqrt{\tau_0})\Omega(N - 1)\sqrt{\tau_L}, \\
\quad a_4 &= (\sqrt{\tau_L} + \hat{A}\sqrt{\tau_0})\Omega\sqrt{\tau_H}.
\end{align*}
\]

From (86) and (90)-(95), we have

\[
E_t[dV(M(t), S(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t))] = -(r - \rho)V(M(t), S(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t))dt. \tag{96}
\]

(96) implies

\[
E_t[e^{-\rho(T-t)}V(M(T), S(T), D(T), \hat{H}_n(T), \hat{H}_{-n}(T))] = e^{-r(T-t)}V(M(t), S(t), D(t), \hat{H}_n(t), \hat{H}_{-n}(t)),
\]

which implies the transversality condition (42) is satisfied if \( r > 0 \).

Q.E.D.

**Recursive Formulas:**

Suppose beliefs have a general form that public signal has precision \( \tau_0 \) and private signal \( i \) has precision \( \tau_i \) \( (i = 1, \ldots N) \). Using equations (25), (24), and (27), we write the recursive formula for the estimate of a growth rate \( G(u) \) as a function of \( G(t) \), where \( u \geq t \),

\[
G(u) = G(t) \cdot e^{-\alpha_G(u-t)} + \sigma_G \cdot \Omega \cdot \sum_{n=0}^{N} \tau_n^{1/2} \cdot \int_{k=t}^{u} e^{-\alpha_G(u-k)} \cdot dB_n^*(k), \tag{97}
\]

where \( dB_n^*(k) = \tau_n^{1/2}/\sigma_G \cdot (G^* - G) \cdot dt + dB_n(k) \). We also use equations (29) and (36) to summarize the dynamics of signals \( H_n(s) \) and \( Z_n(s) \):

\[
H_n(s) = e^{-(\alpha_G + \Omega \tau)(s-t)} \cdot \left( H_n(t) + \int_{u=t}^{s} e^{-(\alpha_G + \Omega \tau)(t-u)} \cdot dI_n(u) \right), \quad s \geq t, \tag{98}
\]

\[
Z_n(s) = e^{-(\alpha_G + \Omega \tau)(s-t)} \cdot \left( Z_n(t) + (\Omega \tau - \Omega \tau) \cdot \int_{u=t}^{s} e^{-(\alpha_G + \Omega \tau)(t-u)} \cdot H_n(u) \cdot du \right). \tag{99}
\]

We next plug in equations (25), (24) and (97) into equation (98) to find the recursive formula for the signal \( H_n(s), n = 0, \ldots N \) at time \( s \) as a function of \( H_n(t) \) and \( G(t) \) at time \( t \), where \( s \geq t \),

\[
H_n(s) = e^{-(\alpha_G + \Omega \tau)(s-t)} \cdot \left( H_n(t) + G(t) \cdot \frac{\tau_n^{1/2}}{\sigma_G} \cdot \frac{e^{\Omega \tau(s-t)} - 1}{\Omega \tau} \right) + B_{h,n}(t, s), \tag{100}
\]

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where term $B_{h,n}^*(t,s)$ aggregates uncertainty expected to unfold between time $t$ and $s$:

$$B_{h,n}^*(t,s) = \int_{u=t}^{s} e^{-(\alpha_G+\Omega\tau)(s-u)} dB_n^*(u) + \tau_n^{1/2} \int_{u=t}^{s} e^{-(\alpha_G+\Omega\tau)(u-k)} dB_n^*(u) du.$$

(101)

We then plug expression for $H_n(u)$ from equation (100) into equation (99) to find the recursive formula for the signal $Z_n(s), n = 0 \ldots N$ at time $s$ as a function of $Z_n(t), H_n(t)$ and $G(t)$ at time $t$, where $s \geq t$,

$$Z_n(s) = e^{-(\alpha_G+\Omega\tau)(s-t)} \cdot \left( Z_n(t) + H_n(t) \cdot (1 - e^{(\Omega\tau-\bar{\Omega}\tau)(s-t)}) + G(t) \cdot \frac{\tau_n^{1/2}(\bar{\Omega}\tau - \Omega\tau)}{\sigma_G \bar{\Omega}\tau} \cdot \left( \frac{e^{\bar{\Omega}\tau(s-t)} - 1}{\Omega\tau} + \frac{e^{(\Omega\tau-\bar{\Omega}\tau)(s-t)} - 1}{(\bar{\Omega}\tau - \Omega\tau)} \right) \right) + B_{z,n}^*(t,s).$$

(102)

where term $B_{z,n}^*(t,s)$ aggregates uncertainty expected to unfold between time $t$ and $s$:

$$B_{z,n}^*(t,s) = e^{-(\alpha_G+\Omega\tau)(s-t)} \cdot (\bar{\Omega}\tau - \Omega\tau) \cdot \int_{u=t}^{s} e^{-(\alpha_G+\Omega\tau)(t-u)} dB_{h,n}^*(t,u) du.$$  

(103)

Q.E.D.

**Formulas of $a_0, a_n, and b_n$:**

Denote the constants $a_n$ and $b_n$, where $n = 0 \ldots N$, as

$$a_0 := \frac{\sigma_G \cdot \tau_n^{1/2} \cdot (\Omega \cdot \tau_0^{1/2}(r + \alpha_G) - C_G \bar{\Omega}\tau_0^{1/2})}{(r + \alpha_D)(r + \alpha_G)} + \frac{\sigma_G \cdot C_G \cdot \bar{\Omega} \cdot \tau_n^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \left( \Omega\tau_0 + \Omega \sum_{n=1}^{N} \tilde{\tau}_n^{1/2} \cdot \tau_n^{1/2} - \bar{\Omega} \cdot \bar{\tau} \right).$$

(104)

$$a_n := \frac{\sigma_G \cdot (\Omega \cdot \tau_n^{1/2}(r + \alpha_G) - C_G \bar{\Omega} \cdot \tilde{\tau}^{1/2} \alpha_G)}{(r + \alpha_D)(r + \alpha_G)} + \frac{\sigma_G \cdot C_G \cdot \bar{\Omega} \cdot \tilde{\tau}}{(r + \alpha_D)(r + \alpha_G)} \left( \tau_n^{1/2} \Omega\tau_0 + \tau_n^{1/2} \Omega \sum_{i=1}^{N} (\tilde{\tau}_i^{1/2} - \bar{\Omega} \cdot \tilde{\tau}) \right).$$

(105)

$$b_n := \frac{\sigma_G \cdot \Omega \cdot \tau_n^{1/2}}{(r + \alpha_D)} + \frac{\sigma_G \cdot C_G \cdot \bar{\Omega} \cdot \tilde{\tau}}{(r + \alpha_D)(r + \alpha_G)} \left( \Omega\tau_0 + \Omega \sum_{n=1}^{N} \tilde{\tau}_n^{1/2} \cdot \tau_n^{1/2} \right).$$

(106)

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