

# Corporate Finance and Risky Inalienable Human Capital\*

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## Abstract

We analyze a dynamic optimal contracting model where the entrepreneur has risky inalienable human capital and investors have limited liability protection. The inabilities for the entrepreneur and investors to commit generate significant distortions for corporate investment, financing, and risk management decisions and influence corporate valuation. We show that corporate risk management and liquidity management are critical components of optimal corporate financial policies. We derive endogenous debt capacity that ties to the entrepreneur's risky inalienable human capital. We show that investment dynamics is highly nonlinear especially when the firm's liquidity is low. Investment distortions via asset sales are critical parts of risk management for firms that are severely financially constrained. Preserving liquidity is thus of the first-order importance to maximize firm value. Additionally, over-investment can be highly valuable for firms to mitigate their concerns to violate limited liability conditions. The fact that we observe firms' excessive investment near bankruptcy may not indicate that equity holders are engaging in excessive risk taking. We also show that equilibrium default on credit line is part of the firm's optimal financial contracting when firms face persistent productivity shocks. Finally, we quantitatively value the net benefits of risk management and liquidity management and find that the values are high.

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# 1 Introduction

An entrepreneur's inability to commit to stay with the firm imposes an important constraint on an entrepreneurial firm's debt capacity and ability to fund investment projects. The first model by Hart and Moore (1994) formalizing this constraint characterizes an entrepreneurial firm's debt capacity and optimal debt repayment path in a highly stylized setting where there is no uncertainty and where the entrepreneur is assumed to consume the proceeds from the investment at the end of the project's life. They recognize that this is a highly restrictive framework but argue that it already yields rich set of results.

In this paper we substantially generalize the Hart and Moore (1994) framework by introducing risk, ongoing consumption and investment, and risk averse preferences. Beyond the obvious benefit of considering the corporate finance problem under inalienable human capital in a substantially less restrictive model, our analysis mainly allows us to study important new aspects of corporate financial policy such as optimal payout (or consumption), investment, liquidity management, and risk management policies. Moreover, our dynamic contracting formulation naturally yields an analytically tractable valuation framework.

We introduce risk in the form of shocks to the capital stock, which affect how profitable investments are. Most interestingly, however, these shocks introduce a stochastic participation constraint for the entrepreneur. The extent to which the entrepreneur is willing to stay with the firm now depends on the history of realized capital shock: if the entrepreneur's human capital is high she must receive a greater promised compensation to be induced to stay. But the entrepreneur is averse to risk and has a preference for smooth consumption. These two opposing issues give rise to a novel dynamic optimal contracting problem between the infinitely-lived risk-averse entrepreneur and fully diversified (or risk-neutral) investors.

We show that the optimal contracting problem can be reduced to a recursive formulation with a single key state variable  $w$ , the ratio between the entrepreneur's *promised* certainty equivalent wealth and her human capital under the contract. The optimal recursive contract for the investor then specifies three state-contingent variables: *i*) the entrepreneur's consumption-capital ratio  $c(w)$ ; *ii*) investment-capital ratio  $i(w)$ , and; *iii*) risk exposure  $x(w)$ . This contract seeks to maximize investors' payoff while providing insurance to the entrepreneur and retaining her. In other words, the optimal contract is a solution to a form

of the well-known tradeoff between risk sharing and incentives in a model of capital accumulation. Here the entrepreneur's dynamic participation constraint at each point in time is in effect her incentive constraint. She needs to be incentivized to stay even when her human capital can be profitably deployed elsewhere.

If the entrepreneur were able to alienate her human capital the optimal contract would simply provide her with a constant flow of consumption and shield her from any risk. Under this contract the firm's investment policy reduces to the familiar Tobin's  $Q$  based policy. But under inalienable human capital the entrepreneur must be prevented from leaving, especially when accumulated capital is high. To retain the entrepreneur in these states of the world the optimal contract must promise her higher wealth and consumption when the capital stock is high, thus exposing her to risk.

We further show that the optimal contract for the investor can be implemented by delegating control over the firm to the entrepreneur in exchange for a credit line with an endogenously determined stochastic limit  $\underline{S}$ . The entrepreneur then maximizes her life-time utility by optimally choosing consumption-capital ratio  $c(s)$ , investment-capital ratio  $\iota(s)$ , and hedge-capital ratio  $\psi(s)$  as a function of entrepreneurs' savings  $s$ . In other words, the optimal contract under risky inalienable human capital can be implemented via debt financing together with cash management and dynamic hedging policies.

The optimal contract provides the entrepreneur with a flat consumption stream as long as the capital stock does not grow too large. When the capital stock increases as a result of investment or positive shocks to the point where the entrepreneur's participation constraint may be violated the contract provides a higher consumption stream to the entrepreneur. Given that the entrepreneur's consumption and wealth are positively correlated with the capital stock under the optimal contract the firm will generally underinvest relative to the benchmark of fully alienable human capital.

For the one-sided commitment problem, we find the following key results. First, the firm's investment is always below the first-best benchmark as the entrepreneur's inalienable human capital constrains the amount of financing that investors can offer to the firm and hence the firm optimally under-invests. As the entrepreneur's performance increases, the firm's financial constraint is relaxed and the firm's investment approaches the first-best benchmark value. Second, the entrepreneur optimally manages the business risk exposure by managing

the credit line and also trading futures or other derivatives to partially hedge the risk exposure. Third, our model (both the contracting formulation and the implementation) provide very natural implications on corporate valuation. We substantially generalize the neoclassical  $q$  theory of investment to allow for limited enforcement and limited liability constraints. We show that the definition of financial slack and/or liquidity can have very different implications on the marginal value of liquidity and marginal value of capital, i.e., marginal  $q$ . Despite different formulations of modern  $q$  theory with financial fictions, both models capture the essence of financial frictions induced by limited enforcement and limited liability. We show that these frictions have substantial impact (conceptually and quantitatively) on corporate investment dynamics.

For the two-sided commitment problem, we have the following striking results. First, the firm may engage in over-investment (compared with the first-best benchmark). The intuition is as follows. In order to make sure that investors do not have incentives to default, the entrepreneur's scaled promised wealth  $w$  cannot be too high otherwise the investors will end up with negative valuation for the enterprise. As a result, the firm needs to substantially increase investment and cut consumption in order to keep the investors stay in the long-term relationship. Despite the excessive investment for firms in bad times (with very low liquidity), this does not imply that equity holders are risk seeking at the expense of debt holders. Second, we find that the entrepreneur's consumption function can be convex when the entrepreneur does sufficiently well again in order to ensure that investors do not walk away from the long-term relationship. Third, we find that risk management policies are highly nonlinear and non-monotonic in these limited enforcement settings. Finally, we find that the quantitative implications of limited liability protection for investors are substantial. For example, in our baseline calibration, the firm's enterprise value is about 20% lower when the firm's limited liability constraint is about to bind.

**Related literature.** A central motivation of the optimal dynamic contracting approach we take here is to make explicit the fundamental tradeoffs that underlie the firm's financial policies. A related motivation is to provide micro-foundations for widely used financial contracts and commonly observed corporate financial policies. Thus, a major insight from the existing literature on financial contracting is that debt is an optimal financial contract

in a wide set of circumstances, such as: i) costly monitoring (Townsend, 1979, and Gale and Hellwig, 1985); ii) limited commitment (Bolton and Scharfstein, 1990 and Hart and Moore, 1994); iii) inalienability of human capital (Hart and Moore, 1994). Another related major result is that a credit line is an optimal financial contract in a dynamic contracting setting with moral hazard (DeMarzo and Fishman, 2007, Biais, Mariotti, Plantin, and Rochet, 2007, DeMarzo and Sannikov, 2006, Biais, Mariotti, Rochet, 2010, and DeMarzo, Fishman, He and Wang, 2012). These results, however, have all been derived under the assumption of risk-neutral preferences for the entrepreneur and investors. By allowing for risk averse entrepreneurs, we not only substantially generalize the results on the optimality of credit lines, risk management (via futures or options or other commonly used derivatives), and debt capacity, but also provide micro-foundations for executive compensation contracts, corporate liquidity management and hedging policies.

There is a growing literature on optimal financial contracting in continuous time (see Sannikov, 2012 for a recent survey) and a few recent papers have considered similar contracting problems to ours. Ai and Li (2013) and Ai, Kiku, and Li (2013) have independently proposed a similar contracting framework to ours to study how investment and managerial compensation vary with firm size and to derive an endogenous size distribution of firms that is consistent with Zipf's law. We focus on the financial implementation with endogenous liquidity accumulation and risk management and also explore risk-return implications. Rampini and Viswanathan (2010, 2013) study corporate risk management and capital structure decisions in a discrete-time dynamic agency model where the contract is subject to limited enforcement and risk neutrality. A key result in their framework is that hedging is not an optimal policy for firms with limited capital that they can pledge as collateral. For such firms, hedging demand in effect competes for limited collateral with investment demand, which tends to be higher for growth firms.

Broadly speaking, our work contributes to the recent and growing literature that studies the implications of dynamic agency on firms' investment and financing decisions. Albuquerque and Hopenhayn (2004), Quadrini (2004) and Clementi and Hopenhayn (2006) study firms' financing and investment decisions under limited commitment. Lorenzoni and Walentin (2007) study Tobin's  $Q$  and investment under limited enforcement. Grochulski and

Zhang (2011) consider a risk sharing problem with limited commitment.<sup>1</sup>

## 2 The model

We consider an optimal long-term contracting problem under limited commitment between an infinitely-lived risk-neutral investor (*the principal*) and a financially constrained, infinitely-lived, risk-averse entrepreneur (*the agent*). The entrepreneur has a proprietary business idea for a growth venture that we represent simply as a production function and a capital accumulation process.

### 2.1 Production Technology

Let  $I$  denote the gross investment by the entrepreneurial firm. We assume that the capital stock  $K$  accumulate as follows:

$$dK_t = (I_t - \delta K_t)dt + \sigma K_t dZ_t, \quad (1)$$

where  $Z$  is a standard Brownian motion,  $\delta \geq 0$  is the expected rate of depreciation and  $\sigma$  is the volatility of a capital depreciation shock.<sup>2</sup>

The firm's capital stock can be interpreted as either tangible capital (property, plant and equipment), the entrepreneur's human capital, or also as firm-specific intangible capital (patents, know-how, brand value, and organizational capital), or any combination of these.

The firm's operating revenue is proportional to its capital stock  $K_t$ . It is given by  $AK_t$ , where  $A$  is constant and measures the firm's productivity. The firm's operating profit  $Y_t$  is then given by:

$$Y_t = AK_t - I_t - \Phi(I_t, K_t), \quad (2)$$

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<sup>1</sup>Kehoe and Levine (1993) is an important early contribution to contracting under limited enforcements. See Ljungqvist and Sargent (2004) for a textbook treatment on these class of models widely used in macro.

<sup>2</sup>As Cox, Ingersoll, and Ross (1985) and the  $AK$ -based endogenous growth literature have found (see Jones and Manuelli, 2005 for a survey), it is analytically simpler to assume that productivity is constant but that the firm is subject to capital depreciation shocks. We consider a version of the model with productivity shocks in an extension.

where the price of the investment good is set to unity and  $\Phi(I, K)$  is the standard adjustment cost function that is at the core of the  $q$  theory of investment.

For simplicity we assume that the firm's adjustment cost  $\Phi(I, K)$  is homogeneous of degree one in  $I$  and  $K$ , and write  $\Phi(I, K)$  in the following homogeneous form:

$$\Phi(I, K) = \phi(i)K, \tag{3}$$

where  $i = I/K$  is the firm's investment-capital ratio and  $\phi(i)$  is an increasing and convex function. As Hayashi (1982) has first shown, with this homogeneity property Tobin's average and marginal  $q$  are equal under perfect capital markets.<sup>3</sup> However, as we will show, under limited contractual enforcement an endogenous wedge between Tobin's average  $q$  and marginal  $q$  will emerge in our model.

## 2.2 Preferences

The infinitely-lived risk-averse entrepreneur has a standard time-additive separable expected utility function over expected positive consumption flows  $\{C_t : t \geq 0\}$  given by:

$$V_t = \mathbb{E}_t \left[ \int_t^\infty \zeta e^{-\zeta(v-t)} U(C_v) dv \right], \tag{4}$$

where  $\zeta > 0$  is the entrepreneur's subjective discount rate and  $U(C)$  is an increasing and concave function, and  $\mathbb{E}_t[\cdot]$  is the time- $t$  conditional expectation. We assume that the entrepreneur has constant relative risk aversion (CRRA) preferences and that  $U(C)$  takes the standard iso-elastic functional form:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}, \tag{5}$$

where  $\gamma > 0$  is the coefficient of relative risk aversion.<sup>4</sup>

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<sup>3</sup>Lucas and Prescott (1971) analyze dynamic investment decisions with convex adjustment costs, though they do not explicitly link their results to marginal or average  $q$ . Abel and Eberly (1994) extend Hayashi (1982) to a stochastic environment and a more general specification of adjustment costs.

<sup>4</sup>Note that since we have normalized the value function with the constant  $\zeta$  in (4), the utility flow in our model is  $\zeta U(C)$  as in various dynamic contracting models.

## 2.3 The Contracting Problem

Without loss of generality we assume that the investor has all the bargaining power and that the contracting game begins at time 0 with the investor making a take-it-or-leave-it long-term contract offer to the entrepreneur, which specifies funding for the venture's initial capital stock  $K_0$ , an investment process  $I = \{I_t : t \geq 0\}$ , and a consumption allocation  $C = \{C_t : t \geq 0\}$  for the entrepreneur, in return for the business income  $\{Y_t : t \geq 0\}$ . The investment and consumption processes can depend on the entire history of output  $Y$  and capital stock  $K$ .

To begin with we assume that the investor can commit to such a long-term contract in perpetuity, but that the entrepreneur cannot commit to run the venture forever. In other words, we assume that the entrepreneur's human capital is inalienable (as in Hart and Moore, 1994) and that the entrepreneur is free to exit the venture at any time. A long-term contract for the entrepreneur is thus akin to an American option that the entrepreneur can exercise at any time. However, in equilibrium, the entrepreneur will not walk away under the optimal contracting. In Section 7, we consider the more general and realistic case of two-sided limited commitments, where the investor is protected by limited liability, and where consequently neither investor nor the entrepreneur can fully commit to continuing under the contract in perpetuity.

We assume that the output process  $Y$  is publicly observable and verifiable. In addition, we assume that the entrepreneur cannot privately save, as is standard in the literature on dynamic moral hazard (see Bolton and Dewatripont, 2005 chapter 10). Under these assumptions the only agency problem that arises in our setting is the entrepreneur's limited-commitment problem.

**The entrepreneur's participation constraint.** When the entrepreneur exits she obtains an outside payoff of  $\widehat{V}(K_t)$  in terms of utils; in other words,  $\widehat{V}(K_t)$  is the entrepreneur's outside value function, which in general depends on accumulated capital  $K_t$  at the moment of exiting from the long-term relationship. Thus, the entrepreneur's dynamic participation/limited-commitment constraint is:

$$V_t \geq \widehat{V}(K_t), \quad t \geq 0. \tag{6}$$

In general, the entrepreneur's outside value function  $\widehat{V}(K_t)$  can be endogenous. We consider a few economic settings to close our model and show that these various settings offer related but distinct economic insights and interpretations for  $\widehat{V}(K_t)$ .

Moreover, at the time of contracting the entrepreneur has a reservation utility level  $\underline{V}_0$ , so that the optimal contract must also satisfy the constraint:

$$V_0 \geq \underline{V}_0. \quad (7)$$

Without loss of generality, we require that  $\underline{V}_0 \geq \widehat{V}(K_0)$  for otherwise the entrepreneur would immediately walk away at time 0. Motivated mostly by our economic insights and also partly for tractability reasons, we assume the following functional form for the outside option payoff:

$$\widehat{V}(K_t) = \frac{1}{1-\gamma} (b\widehat{q}K_t)^{1-\gamma}. \quad (8)$$

where  $\widehat{q}$  is a constant that can be either specified exogenously or determined endogenously, and  $b$  is constant and given by (15). Indeed, under this functional form the entrepreneur's payoff under the optimal contract  $V(K_t)$  can be shown to be homogeneous of degree one in  $K_t$ , which results in a substantial simplification of the optimal contracting solution.

**The optimal contracting problem.** The investor's problem at time 0 is thus to choose dynamic investment and consumption processes  $I$  and  $C$  to maximize the time-0 discounted value of profits,

$$\max_{I, C} \mathbb{E}_0 \left[ \int_0^\infty e^{-rt} (Y_t - C_t) dt \right], \quad (9)$$

subject to the capital accumulation process (1), the production function (2), and the entrepreneur's limited-commitment (6) and the time-0 participation constraint (7).

As we shall show, while the participation (7) constraint is always binding under the optimal contract, the limited-commitment constraints (6) will not bind most of the time.

### 3 The Full-Commitment First-Best Benchmark

Consider to begin with the optimal outcome under full commitment. Our contracting problem then reduces to the neoclassical setting of Hayashi (1982). Indeed, the investor then simply buys off the entire venture from the entrepreneur at time 0 for the reservation utility  $\bar{V}_0$  and takes on all the output risk. The investor then maximizes the present discounted value of the venture's cash flows with respect to  $I$ .

Given the stationarity of the economic environment and the homogeneity of the production technology with respect to  $K$ , there is an optimal time-invariant investment-capital ratio  $i = I/K$  that maximizes the present value of the venture. The following proposition summarizes the main results under full commitment.

**Proposition 1** *The firm's value  $P^{FB}(K)$  is proportional to its capital  $K$ ,  $P^{FB}(K) = q^{FB}K$ , where  $q^{FB}$  is the optimal time-invariant unit value of capital:*

$$q^{FB} = \max_i \frac{A - i - \phi(i)}{r + \delta - i}, \quad (10)$$

and the solution to (10) is the first-best investment-capital ratio, denoted by  $i^{FB}$ .

Note that  $q^{FB}$  is the familiar Tobin  $q$ , which under constant returns to scale is both the marginal and average value of capital, as Hayashi (1982) has shown. Adjustment costs create a wedge between the value of installed capital and newly purchased capital, so that that  $q^{FB} \neq 1$  in general. When the firm is sufficiently productive, that is when  $A > r + \delta + \phi'(0)$ , unit investment is strictly positive,  $i^{FB} > 0$  and  $q^{FB} > 1$ . The expression in (10) represents the present value of a unit of capital under the *Gordon growth formula* for a perpetuity with a constant expected flow income.<sup>5</sup> The unit perpetuity value of capital is thus its expected unit *free cash flow* (cash flow  $A$  minus total capital expenditures, which are given by the sum of investment  $i$  and adjustment costs  $\phi(i)$ ) discounted at the market interest rate  $r$  minus the endogenous growth rate of capital  $i - \delta$ . To ensure that the firm's first-best value

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<sup>5</sup>See DeMarzo, Fishman, He, and Wang (2012) for a very similar present value formulation for Tobin's  $q$  in Hayashi setting. DeMarzo, Fishman, He, and Wang (2012) assume an iid productivity shock in a  $q$ -theoretic setting and we assume that capital stock is subject to stochastic permanent depreciation shocks.

$P^{FB}(K)$  is bounded above we impose the parameter restriction:

$$A < r + \delta + \phi(r + \delta). \quad (11)$$

This inequality (11) ensures that the firm cannot profitably grow faster than the discount rate indefinitely.

We can write (10) equivalently, via the first-order condition (FOC):

$$q^{FB} = 1 + \phi'(i^{FB}), \quad (12)$$

which states that the marginal value of capital, also known as the marginal  $q$ , is equal to the marginal cost of investing,  $1 + \phi'(i)$ , evaluated at the optimum investment  $i^{FB}$ .

Under the full commitment solution the entrepreneur is perfectly insured and obtains a consumption stream that is independent of the firm's investment dynamics:

$$C_t = C_0 e^{-(\zeta - r)t/\gamma}, \quad t \geq 0. \quad (13)$$

That is, consumption changes exponentially at a rate  $(\zeta - r)/\gamma$  per unit of time. To the extent that the investor and entrepreneur have different discount rates,  $\zeta \neq r$ , the optimal contract will be structured so that they can trade consumption intertemporally with each other. Thus, depending on the sign of  $(\zeta - r)$  the entrepreneur's consumption may grow or decline deterministically over time. It is only when the investor and entrepreneur are equally impatient ( $\zeta = r$ ) that the entrepreneur's consumption is constant over time under the optimal full commitment contract.

To complete the solution for the full-commitment case, we now explicitly solve the initial consumption  $C_0$ . First, in Appendix A, we show that for a given level of the entrepreneur's utility  $V$ , we can calculate the corresponding certainty equivalent wealth (CEW) by inverting the expression  $V(W) = U(bW)$  and obtain:

$$W = U^{-1}(V)/b, \quad (14)$$

where  $U^{-1}(\cdot)$  is the inverse function of the constant-relative-risk-aversion utility function

(5) and  $b$  is a normalization constant given by

$$b = \zeta \left[ \frac{1}{\gamma} - \frac{r}{\zeta} \left( \frac{1}{\gamma} - 1 \right) \right]^{\frac{\gamma}{\gamma-1}}. \quad (15)$$

Note that when  $\gamma = 1$ , we have  $b = \zeta e^{\frac{r-\zeta}{\zeta}}$ . Because the entrepreneur's participation constraint (6) at time 0 will always bind due to the investor's optimality, the entrepreneur's reservation utility  $\underline{V}_0$  implies the certainty equivalent wealth  $\underline{W}_0 = U^{-1}(\underline{V}_0)/b$ . And as we show in Appendix A, the entrepreneur's initial consumption  $C_0$  is simply proportional to the CEW  $\underline{W}_0$ , i.e.,

$$C_0 = m \underline{W}_0 = \left( \frac{\zeta}{b} \right)^{\frac{1}{\gamma}} ((1-\gamma)\underline{V}_0)^{\frac{1}{1-\gamma}}, \quad (16)$$

where  $m$  is the marginal propensity to consume (MPC) for the full-commitment case and is given by

$$m = b^{1-\frac{1}{\gamma}} \zeta^{\frac{1}{\gamma}} = r + \gamma^{-1} (\zeta - r). \quad (17)$$

In summary, under perfect capital markets the time-invariance of the firm's technology implies that the first-best investment-capital ratio is constant over time and independent of the firm's history or the volatility and the investor perfectly insures the entrepreneur's consumption smoothing preference. As we will show next, the entrepreneur's inability to fully commit to the venture indefinitely significantly alters these conclusions.

## 4 Optimal Contracting with One-sided Commitment

The first-best outcome is not achievable when the entrepreneur cannot commit to stay. The simple reason is that under the first-best optimal investment policy  $i^{FB}$  the firm's capital stock grows (in expectation) over time. There will therefore be states of the world where  $K_t$  is such that  $\widehat{V}(K_t) > \underline{V}_0$ . Under the first-best contract the entrepreneur would simply walk away at that point. To prevent such an outcome the investor can write a second-best contract where he commits to a consumption flow  $\{C_t : t \geq 0\}$  for the entrepreneur such that  $V_t \geq \widehat{V}(K_t)$ . Since  $\widehat{V}(K_t)$  is a random variable that is growing with  $K_t$  this second-best contract would inevitably expose the entrepreneur to consumption risk. Another way

for the investor to make sure that the entrepreneur will not quit is to let the capital stock grow more slowly by investing less. In short, under one-sided commitment the dynamic contracting problem involves a particular form of the classic agency tradeoff between risk-sharing and incentives. Here the incentives are on the investor side to invest, which have to be traded off against less risk-sharing of consumption risk for the entrepreneur. An important difference, however, with the standard dynamic moral hazard problem is that most of the time the entrepreneur's dynamic participation constraint will not bind. The reason is that if the contract under one-sided commitment were to always hold the entrepreneur down to her participation constraint then the entrepreneur's promised consumption would be inefficiently volatile.

## 4.1 Formulating the optimal contracting problem

The second-best dynamic contracting problem involves making contingent investment  $\{I_t : t \geq 0\}$  and consumption promises  $\{C_t : t \geq 0\}$  to the entrepreneur. Generally, the contracting problem solution is history-dependent. Importantly, we can summarize the history dependence by using the entrepreneur's promised utility  $V$ , as in DeMarzo and Sannikov (2006) and DeMarzo, Fishman, He, and Wang (2012), among others. We can represent the dynamics of the entrepreneur's promised utility process  $V$  defined in (4), as follows:

$$dV_t = \zeta(V_t - U(C_t))dt + x_t V_t dZ_t, \quad (18)$$

where we have changed the variable  $\{C_t : t \geq 0\}$  into a level component  $C_t$  and a volatility component  $\{x_t : t \geq 0\}$  which determines how much the consumption of the entrepreneur under the second-best optimal contract is insured. A higher  $x$  involves less insurance and therefore a higher volatility of the entrepreneur's promised utility  $V$ , and vice-versa for a lower  $x$ . To provide some intuition for the dynamics of (18) we can represent the entrepreneur's expected present discounted payoff under the contract  $\{\widehat{U}_t, t \geq 0\}$  as follows:

$$\widehat{U}_t = \int_0^t e^{-\zeta v} \zeta U(C_v) dv + e^{-\zeta t} V_t = \mathbb{E}_t \left[ \int_0^\infty \zeta e^{-\zeta v} U(C_v) dv \right], \quad (19)$$

so that  $\widehat{U}_t$  is a martingale (under standard regularity conditions). Applying Ito's formula to  $\widehat{U}_t$  and setting its drift to zero we then obtain the dynamics in (18).

By using the promised utility process  $V$ , we make the investor's dynamic contracting problem Markovian with the following two state variables: (1) the entrepreneur's promised utility  $V$  and (2) the venture's capital stock  $K$ . We first characterize the optimization problem in the interior region and then we characterize the boundary conditions.

**The interior region:**  $V > \widehat{V}(K)$ . We may express the investor's optimization problem via the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rF(K, V) = \max_{C, I, x} \left\{ Y - C + (I - \delta K)F_K + \frac{\sigma^2 K^2}{2} F_{KK} + \zeta(V - U(C))F_V + \frac{(xV)^2}{2} F_{VV} + \sigma x K V F_{VK} \right\}. \quad (20)$$

The right side of (20) gives the expected change of the investor's value function  $F(K, V)$ . The first term is the venture's flow profit  $Y - C$  to the investor. The second term reflects the change of the investor's value  $F(K, V)$  resulting from an expected change in the capital stock  $(I - \delta K)$  and the third term represents the change in the investor's value resulting from the volatility of the capital stock. The fourth and fifth terms in turn reflect the change in investor's value from the drift and volatility of the entrepreneur's promised utility  $V$ . Finally, the last term captures how the investor's value is affected by the correlation between  $K$  and  $V$ .<sup>6</sup> By optimally choosing  $C$ ,  $I$ , and  $x$ , the investor earns the rate of return  $r$  at all times by equating the right side of (20) with  $rF(K, V)$  on its left side.

In the interior region, by applying the standard optimality arguments to the HJB equation (20) for  $C$ ,  $I$  and  $x$ , we obtain the following first-order conditions (FOCs):

$$\zeta U'(C) = -\frac{1}{F_V(K, V)}, \quad (21)$$

$$F_K(K, V) = 1 + \Phi_I(I, K), \quad (22)$$

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<sup>6</sup>The variables  $K$  and  $V$  have perfect instantaneous correlation as there is only one exogenous shock (to capital stock  $K$ ) in the model.

and

$$x = -\frac{\sigma K F_{VK}}{V F_{VV}(K, V)}. \quad (23)$$

First, we note that (21) characterizes the entrepreneur's optimal consumption. Intuitively, the marginal utility of consumption  $\zeta U'(C)$  on the left side of (21) equals  $-1/F_V$ , the inverse of the investors' marginal cost of compensating the entrepreneur. Second, (22) characterizes the investors' optimal investment decision. The marginal benefit of investing to the investor,  $F_K(K, V)$ , must equal to  $1 + \Phi_I(I, K)$ , the marginal cost of investing. Finally, (23) characterizes the optimal exposure of the entrepreneur's promised utility  $V$  to the shock of capital accumulation. As we show later,  $x$  is closely tied to the firm's optimal risk management policy.

**The boundary conditions.** At the endogenously determined boundary, the entrepreneur's dynamic participation constraint binds,

$$V = \widehat{V}(K). \quad (24)$$

Along the boundary, the entrepreneur is indifferent between staying within the long-term relationship with the investor and walking away with the outside option. That is if  $V_{t+dt} \geq \widehat{V}(K_{t+dt})$  with probability one, the entrepreneur will never walk away voluntarily. Intuitively, there are two conditions should be satisfied to ensure  $V_{t+dt} \geq \widehat{V}(K_{t+dt})$  with probability one: (1) the drift of  $V_t/\widehat{V}(K_t)$  should be positive (negative) if  $V_t > 0$  ( $V_t < 0$ ) at the boundary  $V_t = \widehat{V}(K_t)$ ; (2) the volatility of  $V_t/\widehat{V}(K_t)$  should be zero at the boundary  $V_t = \widehat{V}(K_t)$ .

As the entrepreneur's promised utility trends to be infinite, i.e.  $V \rightarrow \infty$ , she effectively owns the whole firm and hence has no incentives to walk away. Naturally, the constraint from limited commitment could be ignored, and the investor's value function trends that under first-best case.

In summary, the HJB equation (20), the FOCs (21), (22), and (23), as well as boundary conditions, jointly characterize the solution to the second-best optimal contract.<sup>7</sup> Next, we use our economic insights of the underlying contracting problem to develop an intuitive pro-

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<sup>7</sup>Note that the entrepreneur's dynamic participation constraint must be binding for some  $K$ , for otherwise the investor can always weakly increase his payoff  $F(K, V)$  by lowering the entrepreneur's promised consumption.

cedure that allows us to transform our PDE formulation of the optimal contracting problem into an analytically much more tractable ODE formulation.

## 4.2 Optimal Investment and Investor's Scaled Value Function

A first step in the reduction of the investor's two-dimensional optimization problem to one dimension is a change of state variable from the entrepreneur's promised value  $V$  to the implied entrepreneur's promised (certainty-equivalent) wealth  $W$ , which we simply refer to as promised wealth. As will become clear later this transformation brings out the economics of the optimal contracting problem in a more transparent way. For example, we will show that the entrepreneur's promised wealth  $W$  can be mapped to the liquidity buffer for the firm in Section 5 when analyzing financial implementation.

As in the full-commitment first-best benchmark case, we may equivalently reformulate the investor's value function  $F(K, V)$  to one expressed in terms of  $K$  and  $W$ , which we denote by  $G(K, W)$  as follows,

$$G(K, W) \equiv F(K, V) = F(K, U(bW)), \quad (25)$$

where the second equality follows from  $V(W) = U(bW)$ , the relation between the entrepreneur's utility  $V$  and the promised wealth  $W$ . Applying the Ito's formula to (25), we transform the HJB equation (20) for  $F(K, V)$  into the following HJB equation for  $G(K, W)$ :

$$\begin{aligned} rG(K, W) = \max_{C, I, x} & \left\{ Y - C + (I - \delta K)G_K + \zeta \frac{U(bW) - U(C)}{bU'(bW)} G_W + \frac{\sigma^2 K^2}{2} G_{KK} \right. \\ & \left. + \sigma \frac{xKV}{bU'(bW)} G_{WK} + \frac{(xV)^2}{2} \frac{G_{WW}U'(bW) - G_W bU''(bW)}{U'(bW)^3} \right\}. \end{aligned} \quad (26)$$

And then using the FOCs for  $I$ ,  $C$ ,  $x$  respectively, we obtain

$$1 + \Phi_I(I, K) = G_K, \quad (27)$$

$$U'(bW) = -\frac{\zeta}{b} G_W U'(C), \quad (28)$$

$$x = -\frac{\sigma K G_{WK} b U'(bW)}{V[G_{WW} - G_W b U''(bW)/U'(bW)]}. \quad (29)$$

The second step in the reduction to one dimension is to conjecture and verify that the investor's value function  $G(K, W)$  is homogeneous of degree one in  $K$  and  $W$ , and therefore  $G(K, W)$  can be rewritten as follows:

$$G(K, W) = g(w)K, \quad (30)$$

where  $w = W/K$  is the ratio between the entrepreneur's CEW  $W$  and the firm's capital stock  $K$ . Next, we summarize the valuation equation and boundary conditions for the investor's scaled value function  $g(w)$  in the following proposition.

**Proposition 2** *Under one-sided commitment, the investor's scaled value function  $g(w)$  solves the following ordinary differential equation (ODE),*

$$\begin{aligned} rg(w) = & A + \frac{m\gamma}{1-\gamma} (-g'(w))^{1/\gamma} w + i(w)(g(w) - wg'(w) - 1) - \phi(i(w)) + \frac{\zeta}{1-\gamma} wg'(w) \\ & - \delta(g(w) - wg'(w)) + \frac{\sigma^2 w^2}{2} \frac{\gamma g'(w) g''(w)}{wg''(w) + \gamma g'(w)}, \quad w > \hat{q}. \end{aligned} \quad (31)$$

The ODE (31) is solved subject to the following boundary conditions

$$\lim_{w \rightarrow \infty} g(w) = g^{FB}(w) = q^{FB} - w, \quad (32)$$

$$\lim_{w \rightarrow \hat{q}} \sigma_w(w) = \lim_{w \rightarrow \hat{q}} \frac{\sigma \gamma w g'(w)}{wg''(w) + \gamma g'(w)} = 0. \quad (33)$$

The optimal investment-capital ratio  $i = I/K$ , the entrepreneur's consumption-capital

ratio  $c = C/K$  and risk exposure  $x$  respectively satisfy

$$\phi'(i(w)) = g(w) - wg'(w) - 1, \quad (34)$$

$$c(w) = m(-g'(w))^{1/\gamma} w, \quad (35)$$

$$x(w) = \frac{(1-\gamma)\sigma wg''(w)}{wg''(w) + \gamma g'(w)}. \quad (36)$$

As  $w \rightarrow \infty$ , the entrepreneur effectively owns the whole firm and hence have no incentives to walk away. In this case, the firm chooses the first-best investment and the firm's total value is maximized at  $q^{FB}$  per unit of capital. The left boundary condition (33) implies the volatility of  $w_t$  should be zero at  $w = \hat{q}$  to ensure the entrepreneur don't want to walk away with probability one, and it also gives that  $\lim_{w \rightarrow \hat{q}} g''(w) = \infty$ . The investment FOC (34) effectively equates the marginal  $q$  with the marginal cost of investing  $1 + \phi'(i(w))$  for the financially constrained firm. We will provide detailed analysis later in Section 6. Also note that the incentive problem distorts the investor's ability to offer full insurance to the entrepreneur. The investor optimally trades off the incentive effects against risk sharing by optimally choosing the risk exposure process  $x(w)$  described by (36).

## 5 Implementation: Savings and Risk Management

Having characterized the optimal contract in terms of the scaled promised certainty equivalent wealth  $w = W/K$ , we now provide an intuitive financial implementation of the optimal contract.<sup>8</sup> We show that the optimal contract can be implemented via (1) a standard savings/borrowing account with an endogenously determined stochastic maximal credit limit  $\underline{S}$  and (2) a risk management instrument via futures (or other derivatives such as call options). Given the liquidity and risk management opportunities, the entrepreneur maximizes the life-time utility by optimally choosing dynamic consumption  $C$  and business investment policies  $I$ , saving or borrowing via credit line at the risk-free rate  $r$  up to  $\underline{S}$ , and also using the futures contract to manage the business income risk.

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<sup>8</sup>It is well known that implementation is not unique. We choose an intuitive one and will discuss alternative ways of implementing the dynamic optimal contract.

## 5.1 The Entrepreneur's Optimization Problem

First, we describe the savings/borrowing account. Let  $S_t$  denote the entrepreneur's bank account balance at time  $t$ . At each time  $t$ , the investor imposes a time-varying lower bound  $\underline{S}_t$  on the entrepreneur's bank account balance. That is the following constraint for  $S_t$  must be satisfied

$$S_t \geq \underline{S}_t, \quad (37)$$

where the absolute value  $|\underline{S}_t|$  is the debt capacity that investors would offer to the entrepreneur. Intuitively, we may interpret (37) as a form of collateral constraints. Importantly,  $\underline{S}_t$  will be determined endogenously.<sup>9</sup> As we will show, the credit limit or debt capacity equals the investors' value  $G(K, W)$  at the moment when the entrepreneurs' limited-commitment constraint (6) binds, i.e.,

$$|\underline{S}_t| = G(K_t, \underline{W}_t) = g(\hat{q})K_t. \quad (38)$$

Intuitively, debt capacity  $|\underline{S}_t|$  equals to the investors' value  $G(K, W)$  when the entrepreneur's participating constraint binds. Note that there will be no default in equilibrium and hence debt is risk-free.

Second, we introduce the hedge account via futures.<sup>10</sup> Consider a simple futures position with respect to the underlying capital shock  $dZ$ . With a time- $t$  position of size  $\psi_t K_t$  in the futures hedge account, the instantaneous payoff to the entrepreneur is simply  $\psi_t K_t dZ_t$  over the time increment  $dt$ . Note that the gains and losses of the futures position is instantaneously credited or debited from the bank account and hence there is no default risk. Note that for simplicity, we express the notional position in the futures via  $\psi_t K_t$ . That is,  $\psi_t$  is the hedge position per unit of the firm's capital stock  $K_t$ .

Provided that the credit constraint (37) is satisfied, the entrepreneur could dynamically trade in both the bank and hedge accounts. Therefore, the liquid asset evolves as follows,

$$dS_t = (rS_t + Y_t - C_t)dt + \psi_t K_t dZ_t, \quad S_t \geq \underline{S}_t. \quad (39)$$

<sup>9</sup>See Kiyotaki and Moore (1997) for a model with exogenously imposed collateral constraints.

<sup>10</sup>Bolton, Chen, and Wang (2011) analyze the optimal corporate risk management for a financially constrained firm. In that model, they also analyze the dynamic futures trading strategies.

Note that the drift term  $rS + Y - C$  is the savings rate which is given by the sum of interest income  $rS$  and non-interest income  $Y$  minus consumption  $C$ . Recall that  $Y$  given in (2) equals the difference between output  $AK$  and total cost of investment  $I + \Phi$  including the adjustment costs. The volatility term in (39) captures the effect of hedging via the futures position  $\psi K$  on the entrepreneur's liquidity holdings  $S$ .

In summary, the entrepreneur optimally chooses consumption  $C$ , investment  $I$ , and futures position  $\psi$  to maximize utility given by (4)-(5) subject to the liquidity accumulation dynamics (39) and the endogenous borrowing limit (37), where  $\underline{S}$  is specified in (38). Next, we solve the entrepreneur's optimization problem.

As we will show next, the entrepreneur's optimization problem stated above yields the same resource allocation outcome as the optimal contracting problem does. That is, this entrepreneur's optimization problem implements the optimal dynamic contract. We use  $J(K, S)$  to denote the entrepreneur's value function.

## 5.2 Solution

First, we use the HJB equation to characterize the entrepreneur's optimality in the interior region, i.e.,  $S > \underline{S}$ , the entrepreneur has some un-used financial slack. The value function  $J(K, S)$  satisfies the following HJB equation:

$$\begin{aligned} \zeta J(K, S) = \max_{C, I, \psi} & \zeta U(C) + (I - \delta K)J_K + (rS + AK - I - \Phi(I, K) - C)J_S \\ & + \frac{\sigma^2 K^2}{2} J_{KK} + \sigma K^2 \psi J_{KS} + \frac{K^2 \psi^2}{2} J_{SS}. \end{aligned} \quad (40)$$

At the boundary  $\underline{S}$ , the entrepreneur's participation constraint binds, in that

$$J(K, \underline{S}) = \widehat{V}(K). \quad (41)$$

As for the optimal contracting problem, we have the homogeneity property for the implementation formulation. Guided by economic insights, we express the entrepreneur's value

function  $J(K, S)$  as follows,

$$J(K, S) = \frac{(bH(K, S))^{1-\gamma}}{1-\gamma} = \frac{(bh(s)K)^{1-\gamma}}{1-\gamma}, \quad (42)$$

where  $s = S/K$ . Here, the entrepreneur's certainty equivalent wealth  $H(K, S)$  can be written as  $H(K, S) = h(s)K$  and the normalization constant  $b$  is given by (15), the same constant as that in (14) for the standard consumption-savings problem.

The following proposition summarizes the model solution for the optimal financial implementation.<sup>11</sup>

**Proposition 3** *The scaled value  $h(s)$  solves the following ODE,*

$$\begin{aligned} 0 = & \frac{h(s)}{1-\gamma} \left[ \gamma m h'^{\frac{\gamma-1}{\gamma}} - \zeta \right] - \delta h(s) + [(r + \delta)s + A] h'(s) \\ & + \iota(s)(h(s) - (s + 1)h'(s)) - \phi(\iota(s))h'(s) - \frac{\gamma\sigma^2}{2} \frac{h(s)^2 h''(s)}{h(s)h''(s) - \gamma h'(s)^2}, \end{aligned} \quad (43)$$

subject to the following boundary conditions:

$$h(\underline{s}) = \hat{q}, \quad (44)$$

$$\lim_{s \rightarrow \underline{s}} \sigma_s(s) = \lim_{s \rightarrow \underline{s}} \frac{\sigma \gamma h(s) h'(s)}{h(s) h''(s) - \gamma h'(s)^2} = 0, \quad (45)$$

$$\lim_{s \rightarrow \infty} h(s) = q^{FB} + s. \quad (46)$$

Consumption  $c(s) = C/K$ , investment  $\iota(s) = I/K$ , and hedging policies  $\psi(s)$  satisfy

$$c(s) = mh(s)h'(s)^{-1/\gamma}, \quad (47)$$

$$\phi'(\iota(s)) = \frac{h(s)}{h'(s)} - s - 1, \quad (48)$$

$$\psi(s) = \sigma \frac{sh''(s)h(s) + \gamma h'(s)(h(s) - sh'(s))}{h(s)h''(s) - \gamma h'(s)^2}. \quad (49)$$

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<sup>11</sup>Wang, Wang, and Yang (2012) solve an entrepreneur's optimal consumption-savings, business investment, and portfolio choice problem with endogenous entry and exit decisions. By exploiting homogeneity, they derive the optimal investment policy in a  $q$ -theoretic context with incomplete markets. In our model, we optimally implement the solution of the optimal contacting problem.

Here,  $s$  is the entrepreneur's savings  $S$  scaled by  $K$ . ODE (43) characterizes the entrepreneur's scaled certainty equivalent wealth  $h(s)$ . Equation (44) states that as the entrepreneur exhausts the debt capacity (as  $s \rightarrow \underline{s}$ ),  $h(s)$  approaches the entrepreneur's scaled outside option  $\hat{q}$ . Additionally, to ensure that the entrepreneur does not walk away (or equivalently does not default on debt), the volatility of  $s$  shall approach zero, stated in (45), and the drift  $\mu_s(\underline{s}) > 0$  to ensure that the constraint is satisfied at all times. The left condition (45) implies  $\lim_{s \rightarrow \underline{s}} h''(s) = -\infty$ . Finally, as  $s \rightarrow \infty$ , the entrepreneur can self insure, markets are effectively complete for the entrepreneur and hence the entrepreneur's scaled promised  $w$  approaches  $q^{FB} + s$ , the sum of the first-best  $q$  and its slack  $s$ , as stated in (46). In this case, the firm chooses the first-best investment and the value of capital stock is maximized at  $q^{FB}$  per unit of capital.

Equation (47) gives the optimal consumption rule. Note that the MPC in this case is given by  $mh'(s)^{-1/\gamma}$  which is lower than the first-best MPC  $m$  as the financial constraint implies  $h'(s) > 1$ . We will discuss the FOC for investment (48) in detail in the next section. Finally, (49) gives the optimal hedging policy  $\psi(s)$ .

## 6 Analysis

### 6.1 Parameter choices.

For illustrational simplicity, we consider the following quadratic adjustment cost function:

$$\phi(i) = \frac{\theta i^2}{2}, \quad (50)$$

where the parameter  $\theta$  measures the degree of the adjustment cost. A higher value  $\theta$  implies a more costly adjustment process. For this case, we have explicit formulas for Tobin's  $q$  and the optimal investment-capital ratio  $i$ :

$$q^{FB} = 1 + \theta i^{FB}, \quad \text{and} \quad i^{FB} = r + \delta - \sqrt{(r + \delta)^2 - 2 \frac{A - (r + \delta)}{\theta}}. \quad (51)$$

When applicable, all parameter values are annualized. The risk-free interest rate is

$r = 5\%$ . The subjective discount rate is set to equal to the risk-free rate,  $\zeta = r = 5\%$ . Under the first-best setting, consumption is constant at all times. We choose the widely used value for the coefficient of relative risk aversion,  $\gamma = 2$ . On the real investment side, we rely on the findings of Eberly, Rebelo, and Vincent (2009) who provide empirical evidence in support of Hayashi (1982). Following their work, we set the expected productivity  $A = 20\%$  and the volatility of productivity shocks  $\sigma = 20\%$ . Fitting the complete-markets  $q^{FB}$  and  $i^{FB}$  to the sample averages, we set the adjustment cost parameter as  $\theta = 2$  and the rate of depreciation for capital stock as  $\delta = 12.5\%$ .<sup>12</sup> Given these baseline parameters we have the following outputs from our model the first-best market-to-book ratio is  $q^{FB} = 1.2$  and the first-best investment-capital ratio is  $i^{FB} = 0.1$ .

We next complete our calibration by choosing the entrepreneur's outside option. We can either choose an exogenously specified  $\hat{q}$  or take a stand on the endogenous determinant of the entrepreneur's outside option. In our baseline calibration, we endogenize the outside option  $\hat{q}$  by considering the entrepreneur's value function under autarky. Suppose that the entrepreneur could walk away with 100% of her human capital but afterwards will be permanently shut out of capital markets. So the penalty here is the forgone consumption smoothing opportunities. As we show in the appendix, with this outside option specification, we have

$$\hat{q} = \frac{(\zeta(1 + \phi'(\hat{i}))(A - \hat{i} - \phi(\hat{i}))^{-\gamma})^{\frac{1}{1-\gamma}}}{b}, \quad (52)$$

where the optimal investment-capital ratio,  $\hat{i}$ , solves the following equation:

$$\zeta = \frac{A - \hat{i} - \phi(\hat{i})}{1 + \phi'(\hat{i})} + (\hat{i} - \delta)(1 - \gamma) - \frac{\sigma^2 \gamma (1 - \gamma)}{2}. \quad (53)$$

Using parameter values for our baseline case summarized in Table 1, we obtain  $\hat{q} = 0.41$ , which implies that the entrepreneur will voluntarily walk away if the promised scaled wealth  $w$  falls below  $\hat{q} = 0.41$ . Admittedly, imposing autarky for the entrepreneur is only one form of penalty when the entrepreneur reneges on the contract. Later in the paper, we also consider alternative ways to endogenize  $\hat{q}$ .

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<sup>12</sup>The averages are 1.2 for Tobin's  $q$  and 0.1 for the investment-capital ratio, respectively, for the sample used by Eberly, Rebelo, and Vincent (2009). The imputed  $\theta = 2$  is in the range of estimates used in the literature. See Whited (1992), Hall (2004), Riddick and Whited (2009), and Eberly, Rebelo, and Vincent (2009).

Table 1: SUMMARY OF PARAMETERS

This table summarizes the parameter values for the baseline model.

Parameters	Symbol	Value
Risk-free rate	$r$	5%
The entrepreneur's discount rate	$\zeta$	5%
The entrepreneur's relative risk Aversion	$\gamma$	2
Capital depreciation rate	$\delta$	12.5%
Volatility of capital depreciation shock	$\sigma$	20%
Quadratic adjustment cost parameter	$\theta$	2
Firm's productivity	$A$	20%

## 6.2 Financial Slack $S$ and Promised Wealth $W$

There are two liquidity measures in our model: financial slack  $S = sK$  in the financial implementation formulation and the entrepreneur's promised wealth  $W = wK$  in the contracting problem. These two liquidity variables are related to each other as follows:

$$s = -g(w), \quad w = h(s). \quad (54)$$

Intuitively, the higher the entrepreneur's scaled promised wealth  $w$ , the "looser" the entrepreneur's enforcement constraint, and the less distortionary the investment decisions are. Indeed, as the investors' value  $g(w)$  decreases with  $w$ , financial slack  $s = -g(w)$  increases with promised wealth  $w$ , consistent with our intuition that the higher the promised wealth  $w$ , the more financial slack that the firm has. Also based on the implementation, the entrepreneur's certainty equivalent wealth  $w$  is simply given by  $h(s)$ . Therefore, the composition of  $-g$  and  $h$ , denoted by  $-g \circ h$ , yields the identity function, i.e.  $-g(h(s)) = s$ .

In Figure 1, we plot  $s = -g(w)$ , sensitivity  $s'(w) = -g'(w)$ ,  $w = h(s)$ , and its sensitivity  $w'(s) = h'(s)$ . Panel A shows that  $s = -g(w)$  is increasing in  $w$  and the marginal value  $s'(w)$  is also increasing and approaches unity as  $w \rightarrow \infty$ . Intuitively, the higher the entrepreneur's payoff  $w$ , the less likely that the entrepreneur will walk away and hence the higher the value of slack in the long-term bilateral relationship. Additionally, the marginal value of  $w$  decreases as  $w$  increases implying that  $s$  is concave in  $w$ . At the left boundary where  $w = \hat{q} = 0.41$ , the

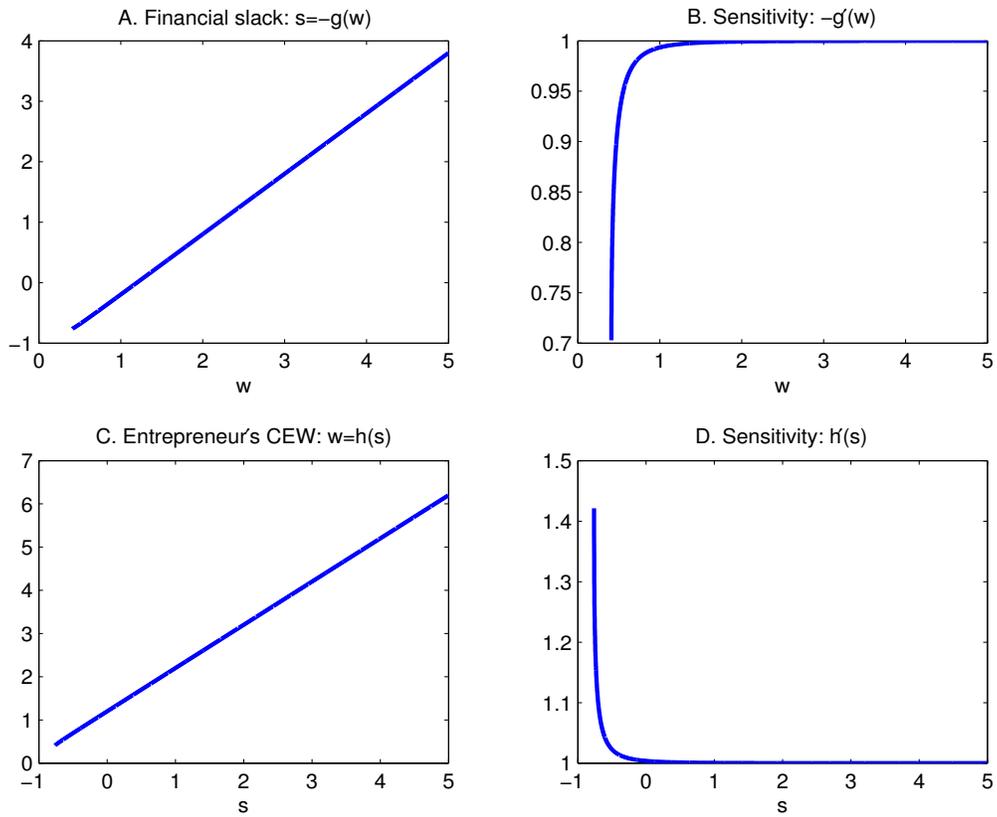


Figure 1: **Scaled financial slack**  $s = -g(w)$ , **scaled CEW**  $w = h(s)$ , and their **sensitivities**. Note that  $-g(h(s)) = s$ .

entrepreneur's financial slack reaches the lower boundary  $\underline{s} = -0.764$ , which is the maximal amount of risk-free credit that can be extended to the entrepreneur in this implementation, we also refer to  $-\underline{s}$  as the debt capacity, which equals 0.764 in our baseline case.

Panel C of Figure 1 plots the entrepreneur's promised wealth  $w$  as a function of the scaled financial slack  $s$ . Intuitively, the higher the value of financial slack  $s$ , the less likely that the entrepreneur will walk away and hence the higher the value of  $w$  in the long-term bilateral relationship. At the left boundary where  $\underline{s} = -0.764$ , the entrepreneur is deeply in debt. That is, the maximal size of the risk-free credit line that can be extended to the entrepreneur is  $-\underline{s} = 0.764$ , which is the debt capacity (per unit of capital). Panel B plots the sensitivity of  $s$  with respect to  $w$ . Note that  $s'(w) = -g'(w)$  is increasing in  $w$  which implies that  $s$  is concave in scaled CEW  $w$ . Additionally,  $s'(w) \leq 1$  and  $s'(w)$  approaches unity as  $w \rightarrow \infty$ .

### 6.3 Marginal values of liquidity and slack

Now we use the properties of  $g(w)$  to derive implications for  $q$ . Total firm value, the sum of the principal's benefit and the promised utility to the entrepreneur, is

$$P(K, W) = G(K, W) + W. \quad (55)$$

The average  $q$ , which is the ratio between firm value and capital stock and given by

$$p(w) = \frac{P(K, W)}{K} = g(w) + w. \quad (56)$$

Notice that this definition of average  $q$  is consistent with the definition of  $q$  in the first-best benchmark (Hayashi, 1982). And marginal value of liquidity, which measures the incremental impact of a unit of promised wealth  $W$  on firm value, and it is equate to the sum of one and the marginal value of CEW to principal's benefit, that is

$$P_W(K, W) = p'(w) = g'(w) + 1. \quad (57)$$

Next, we turn to the implementation version of our model. We define the *enterprise*

value as entrepreneur's certainty equivalent wealth minus liability/cash, i.e.

$$E(K, S) = H(K, S) - S. \quad (58)$$

And the corresponding average  $q$  is the ratio between enterprise value and capital stock and given by

$$e(s) = \frac{E(K, S)}{K} = h(s) - s. \quad (59)$$

Again, as  $s \rightarrow \infty$ , financial frictions become irrelevant and the firm's enterprise value  $e(s)$  approaches the first-best value,  $q^{FB}$ . And the corresponding marginal value of slack is

$$E_S(K, S) = e'(s) = h'(s) - 1. \quad (60)$$

As we will show, the enterprise value increases as  $s$  increases, i.e.,  $e'(s) > 0$ .

Because corporate liquidity  $W$  and financial slack  $S$  are valuable for the firm beyond its face value, the firm's enterprise value depends on  $W$  and/or  $S$ . Under the MM condition, the firm/enterprise value is independent of the liquidity/financial slack, and we have  $P^{FB}(K, W) = E^{FB}(K, S) = q^{FB}K$ . Panels A and B of Figure 2 plot the firm's value losses due to the entrepreneur's limited enforcement, measured by  $q^{FB} - p(w)$  and  $q^{FB} - e(s)$ , respectively. And Panels C and D of Figure 2 plot the marginal value of liquidity  $P_W(K, W)$  and the marginal value of savings  $E_S(K, S)$ , respectively. At the left boundary  $w = \hat{q}$ , the value loss is  $q^{FB} - p(\hat{q}) = q^{FB} - e(\underline{s}) = 0.027$ . Despite being risk neutral, the investor effectively behaves in a risk-averse manner due to the entrepreneur's limited enforcement, as we can see from the concavity of the investor's scaled value  $p(w)$ . This concavity property is critical in our agency model and fundamentally differentiates our model from the neoclassical Hayashi (1982) result where volatility has no effect on firm value. For sufficiently high value of  $w$ , the effects of limited enforcement is weak and eventually vanishes as  $w \rightarrow \infty$ . Note that the entrepreneur's scaled certainty equivalent wealth for the enterprise,  $e(s) = h(s) - s$ , is also concave in savings  $s$ . Quantitatively, we see that the marginal value of liquidity  $P_W(K, W)$  and the marginal value of savings  $E_S(K, S)$  become very high as the firm exhausts its liquidity  $W$  or savings. For example,  $P_W(K, \hat{q}K) = p'(\hat{q}) = 0.30$  and  $E_S(K, \underline{s}K) = e'(\underline{s}) = 0.42$  at the left boundary  $\hat{q} = 0.41$  and  $\underline{s} = -0.764$ .

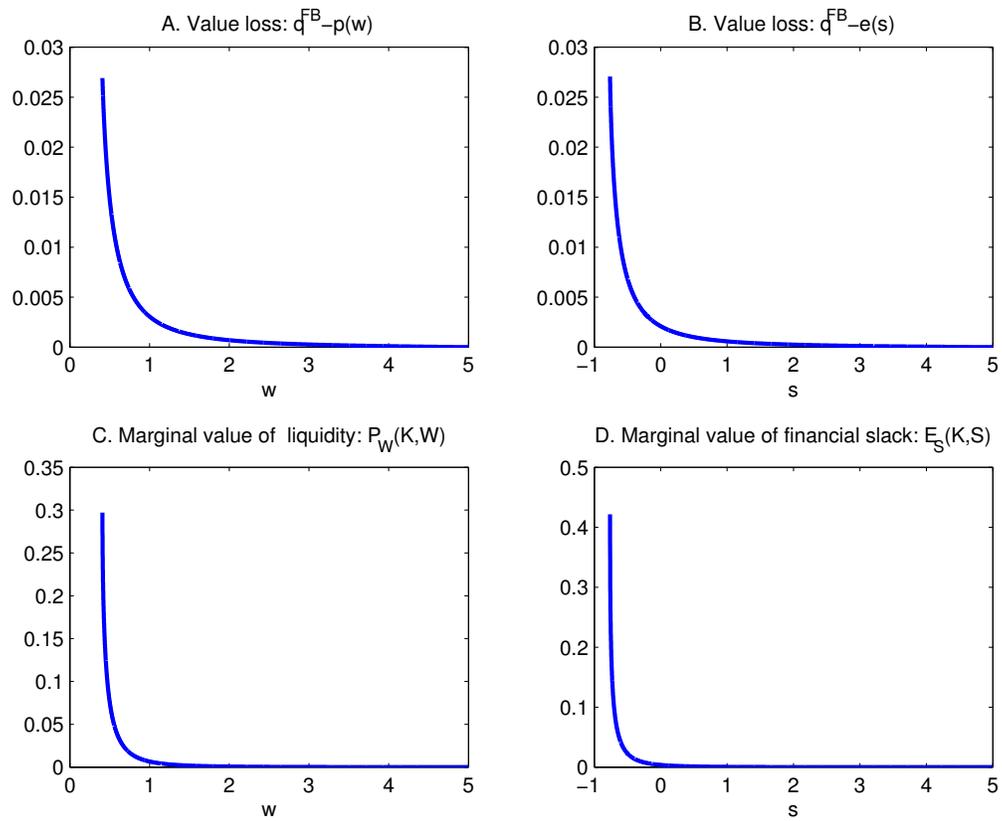


Figure 2: Value losses, the marginal value of liquidity  $P_W(K, W)$  and the marginal value of financial slack  $E_S(K, S)$ .

## 6.4 Tobin's average $q$ , Marginal $q$ and Investment

In this subsection, we focus on corporate investment and valuation implications of our model. Following the macro investment literature, we define marginal value of capital, which is also referred to as marginal  $q$ , as the marginal increase of firm value as the firm increases its capital stock  $K$  by a unit. Using our contracting formulation, we have

$$P_K(K, W) = p(w) - wp'(w) = g(w) - wg'(w). \quad (61)$$

Similarly, using our implementation, we may define the marginal value of capital, and equivalently marginal  $q$ , as

$$E_K(K, S) = e(s) - se'(s) = h(s) - sh'(s). \quad (62)$$

With convex capital adjustment costs and under perfect capital markets, marginal  $q$  is sufficient for us to infer the firm's dynamics. However, empirically marginal  $q$  is not observable. It is often easier to measure average  $q$  despite potential measurement errors. For these reasons, average  $q$  is often used in empirical studies as a proxy for the firm's investment opportunities.

Under full commitment, marginal  $q$  equals average  $q$  in our model as in Hayashi (1982) due to the homogeneity property. However, limited enforcement causes the marginal value of capital,  $P_K(K, W) = p(w) - wp'(w)$ , to differ from the average value of the capital stock,  $p(w)$ . Moreover, we show that average  $q$  is greater than marginal  $q$

$$p(w) > P_K(K, W) = p(w) - wp'(w), \quad (63)$$

as  $w > 0$  and the marginal value of promised wealth is positive,  $p'(w) > 0$  due to limited enforcement as shown in Figure 2.

Panel A of Figure 3 shows that both Tobin's average  $q$  and marginal  $q$  are increasing in  $w$  as the firm's value  $p(w)$  is increasing and concave. Moreover, the wedge between average and marginal  $q$ ,  $p(w) - P_K(K, W) = wp'(w)$  is monotonically decreasing in  $w$ , and the wedge  $p(w) - P_K(K, W) = wp'(w)$  reaches the maximum at the boundary  $w = \hat{q}$ . Our model's

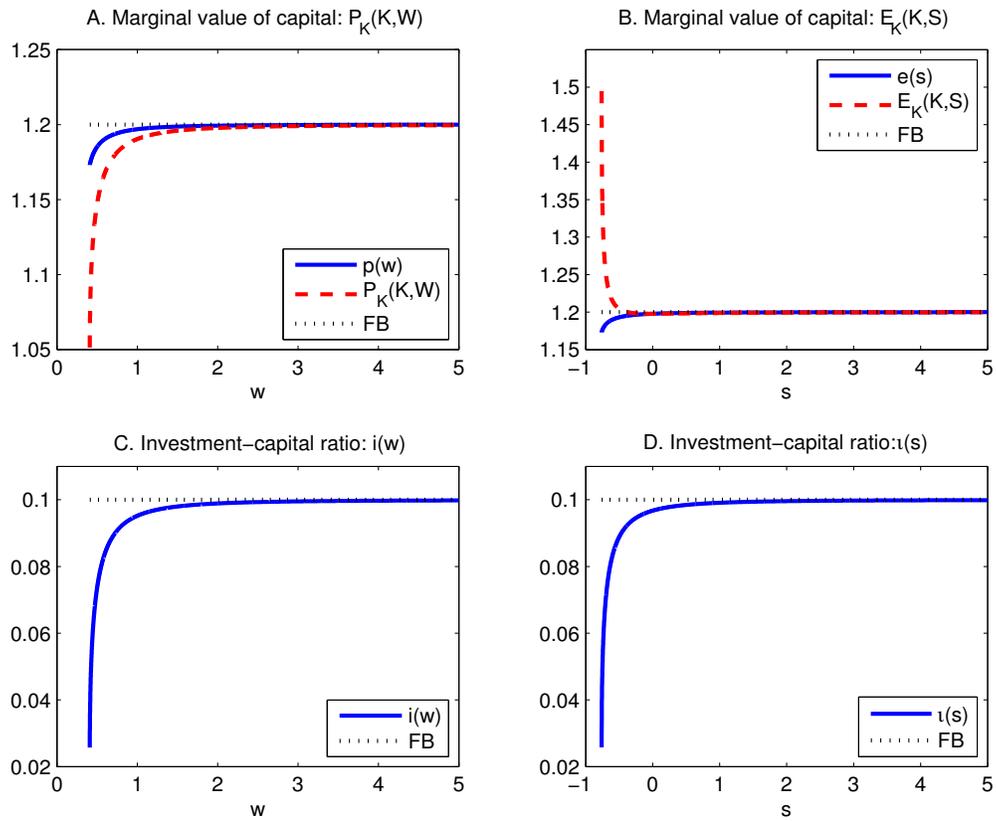


Figure 3: Marginal  $q$ , average  $q$ , and the investment-capital ratio  $i$ .

prediction on this wedge (being monotonic in  $w$ ) differs from the prediction that this wedge is non-monotonic in DeMarzo, Fishman, He, and Wang (2012), (henceforth DFHW), as the entrepreneur in our model does not walk away in equilibrium while the entrepreneur will be fired as the firm runs out of its liquidity  $w$  in DFHW.

Due to the concavity of  $p(w)$ , investment is increasing with  $w$  and approaches  $i^{FB}$  as  $w \rightarrow \infty$ , as we see from Panel C. Using the FOC (34) and the fact that capital adjustment cost  $\phi(\cdot)$  is convex, we have

$$i'(w) = -\frac{1}{\phi''(w)}wg''(w) > 0. \quad (64)$$

With one-sided commitment, we obtain an unambiguous under-investment result. However, as we will show, with two-sided limited commitments, the firm may either under- or over-invest.

Now we turn to the model's predictions on Tobin's average  $q$ , marginal  $q$ , and investment in our implementation formulation. Surprisingly, marginal  $q$  in this formulation is decreasing in savings  $s$ . Indeed, the marginal  $q$  approaches  $E_K(K, \underline{s}K) = 1.5$  at the left boundary  $\underline{s} = -0.764$ , which is much higher than average  $q$ ,  $e(\underline{s}) = 1.17$ . Panel B of Figure 3 plots both  $e(s)$ , the firm's average  $q$ , and  $E_K(K, sK)$ , the firm's marginal  $q$ . The conventional wisdom is that a firm with a high marginal  $q$  invests more, *ceteris paribus*. Here it is not the case due to the fact that the financing cost is endogenously higher when marginal  $q$  is high. See Panel D where  $\iota(s)$  is monotonically increasing with  $s$  and is lower than the first-best investment level  $i^{FB} = 0.1$ .

How do we reconcile a high marginal  $q$  measured by  $E_K(K, sK)$  with underinvestment, that is,  $\iota(s) < i^{FB}$ ? The key missing component here is the marginal value of liquidity  $e'(s)$ . We may rewrite the investment FOC (48) in our implementation formulation as

$$1 + \phi'(\iota(s)) = \frac{h(s) - sh'(s)}{h'(s)} = \frac{E_K(K, sK)}{e'(s) + 1}. \quad (65)$$

This investment equation states that the marginal cost of investing (on the left side) equals the ratio between  $E_K = H_K$ , the marginal  $q$  in the numerator on the right side of (65), and the marginal value of liquidity  $h'(s) = e'(s) + 1$  in the denominator. Intuitively, the firm's

marginal value of liquidity  $e'(s)$  is high precisely when the firm's marginal  $q$  measured by  $E_K$  is high. This correlation explains why corporate investment is below the first-best level.

Limited enforcement frictions have quantitatively significant effects on corporate investment especially when corporate liquidity is low. For example, at the left boundary  $w = \hat{q} = 0.41$  and equivalently  $s = -0.764$ ,  $i = 0.02$ , which is much lower than  $i^{FB} = 0.1$ . Again, as  $w \rightarrow \infty$ , limited enforcement constraints become irrelevant and the firm's investment achieves the first-best level  $i^{FB}$ .

## 6.5 Consumption

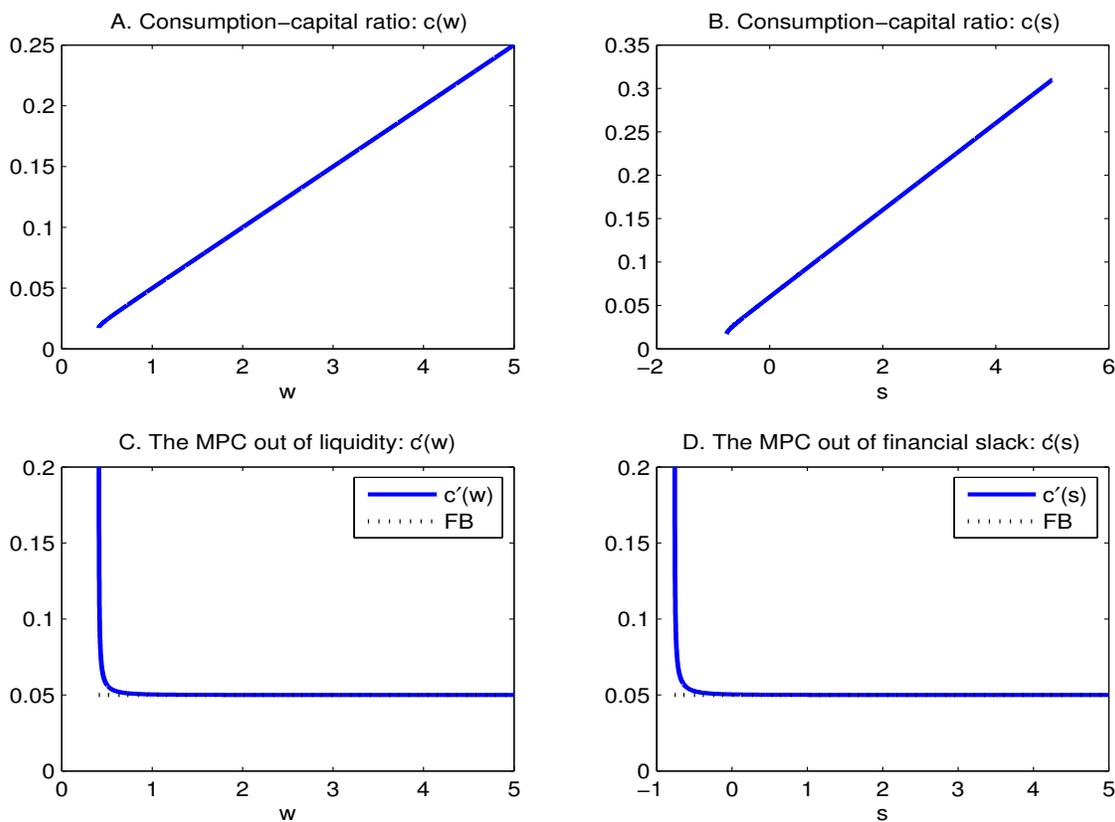


Figure 4: **The consumption and MPC of liquidity/financial slack.**

We now turn to the optimal consumption policies in the two equivalent formulations. Panels A and C of Figure 4 plot the consumption-capital ratio as a function of scaled promised wealth  $w$  and a function of scaled savings  $s$ , respectively. To highlight the curvature of the

consumption function clearly, we also plot the MPCs in Panels C and D of Figure 4, which clearly show that the MPC under limited enforcement is *higher* than the first-best MPC,  $c'(w) > m$  and  $c'(s) > m$ , where  $m = 5\%$  is the first-best MPC in our calibration. Again, due to the limited enforcement constraint, liquidity is valuable for both investment and consumption smoothing,  $P_W = p'(w) > 0$  and  $E_S = e'(s) > 0$ . Moreover, the lower the liquidity  $w$  or savings  $s$ , the greater the MPC is as the entrepreneur is much more constrained. We thus have a concave consumption function induced by the limited enforcement friction. The optimal concave consumption function in our optimal contracting framework is consistent with the concave consumption function in standard incomplete-markets environments (see Carroll and Kimball, 1996). Note that the MPC near  $\hat{q} = 0.41$  or equivalently  $\underline{s} = -0.764$  will be infinite due to  $\lim_{w \rightarrow \hat{q}} g''(w) = \infty$  and  $\lim_{s \rightarrow \underline{s}} h''(s) = -\infty$ .

In our model with one-sided commitment, investors do not walk away even when their valuation of the business turns negative as investors are assumed to fully commit *ex ante*. While *ex ante* valuable to fully commit *ex post*, this outcome is clearly *ex post* sub-optimal from investor's perspectives. Indeed, it is inconsistent with the standard limited liability protection for investor in publicly traded firms, limited liability corporations, or limited partnerships. In the next section, we generalize our dynamic contracting model by allowing for two-sided limited commitments. That is, investor cannot commit not to walk away from the long-term contracting relationship. Under this more plausible setting, we derive novel and intuitive results that we could not have otherwise obtained with one-sided commitments. For example, we find that it may be optimal for the firm to over-invest (relative to the first-best investment benchmark) in order to keep the investor from not walking away.

## 7 Two-Sided Limited Commitments

In this section, we consider the case where investor cannot commit to *ex post* negative net present value projects. In this case, feasibility requires policies  $C, I$  and  $x$  to satisfy

$$F(K_t, V_t) = \max_{C, I, x} \mathbb{E}_t \left[ \int_t^\infty e^{-r(v-t)} (Y_v - C_v) dv \right] \geq 0, \quad (66)$$

as well as the entrepreneurs' limited commitment constraint  $V_t \geq \widehat{V}(K_t)$  for all  $t$ . We refer to this new contract design problem as one with two-sided limited commitments.

## 7.1 Model and solution

We first summarize the solution for the optimal contracting problem and then provide a financial implementation for the contracting model solution.

### 7.1.1 Optimal contracting

Let  $\overline{V}(K)$  denote the entrepreneur's maximal promised utility ensuring that the investor will not voluntarily walk away. That is  $\overline{V}(K_t) \geq V_t \geq \widehat{V}(K_t)$  must be held under two-sided limited commitments. By using the change of variable, we may express the region for admissible promised wealth  $W$  via  $\overline{W}(K_t) \geq W_t \geq \widehat{q}K_t$  where the upper bound for the entrepreneur's promised wealth  $\overline{W}(K_t)$  is given by  $\overline{W}(K) = U^{-1}(\overline{V}(K))/b$ .

Recall that under one-sided limited commitment, to ensure that the limited enforcement constraint  $V_{t+dt} \geq \widehat{V}(K_{t+dt})$  is satisfied with probability one, we need to require that the volatility process of  $V_t/\widehat{V}(K_t)$  approaches zero at the boundary  $V_t = \widehat{V}(K_t)$ . Moreover, the drift of  $V_t/\widehat{V}(K_t)$  should move the process towards the interior region.<sup>13</sup> Following essentially the same analysis, we require that the volatility of  $V_t/\overline{V}(K_t)$  approaches zero at both the lower boundary  $\widehat{V}(K_t)$  and the lower boundary  $\overline{V}(K_t)$  to ensure  $V_{t+dt} \leq \overline{V}(K_{t+dt})$  with probability one for the entire interior region.<sup>14</sup> Under these conditions, neither the investor and the entrepreneur will walk away from the long-term contract.

With these boundary conditions, we obtain the HJB equation for the investor's value function  $F(K, V)$  as we have for the case with one-sided limited commitment. That is, the investors' limited liability constraint only changes the boundary condition for the upper boundary  $\overline{V}(K)$ . By using the fact that the investor's value equals zero at the upper

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<sup>13</sup>Specifically, the drift of  $V_t/\widehat{V}(K_t)$  should be positive if  $V_t > 0$  (for the case with  $\gamma > 1$ ) and negative if  $V_t < 0$  (for the case with  $\gamma > 1$ ).

<sup>14</sup>Similarly, we require that the drift of  $V_t/\overline{V}(K_t)$  at both boundaries pull the process into the interior region.

boundary  $\bar{V}(K)$  and making the change of variable from  $V$  to  $W$  via  $V = U(bW)$ , we obtain

$$G(K_t, \bar{W}(K_t)) = F(K_t, \bar{V}(K_t)) = 0. \quad (67)$$

Using the homogeneity property, we also have

$$g(\bar{w}) = 0, \quad (68)$$

where  $\bar{w} = \bar{W}(K)/K$ . As we have discussed earlier, we require that  $\sigma_w(w)$ , the volatility for  $w$ , approaches zero at  $\bar{w}$ ,

$$\lim_{w \rightarrow \bar{w}} \sigma_w(w) = \lim_{w \rightarrow \bar{w}} \frac{\sigma \gamma w g'(w)}{w g''(w) + \gamma g'(w)} = 0, \quad (69)$$

which implies

$$\lim_{w \rightarrow \bar{w}} g''(w) = -\infty. \quad (70)$$

In summary, for the case with two-sided limited commitments, investor's scaled value function  $g(w)$  solves ODE (31) subject to (33), the boundary condition at  $\hat{q}$ , and the following new boundary conditions at the endogenously chosen boundary  $\bar{w}$ ,

$$g(\bar{w}) = 0, \quad (71)$$

$$\lim_{w \rightarrow \bar{w}} g''(w) = -\infty. \quad (72)$$

As we will show, the new boundary conditions (71)-(72) at the endogenously determined boundary  $\bar{w}$  will have important implications on liquidity holding, risk management, and corporate investment decisions.

### 7.1.2 Implementation: Credit and hedging

Next we provide a financial implementation of the optimal contract for the case of two-sided limited commitments. As we have shown for the one-sided commitment case, in implementation, the entrepreneur's savings in the bank account  $S_t$  can be expressed via

$S_t = -G(K_t, W_t)$ . With the limited-liability condition for investors, we have  $S_t \leq 0$ . Combining with the endogenous debt capacity due to the entrepreneur's limited commitment, we obtain

$$0 \geq S_t \geq \underline{S}_t. \quad (73)$$

In this implementation, the entrepreneur always borrows,  $S_t < 0$ , and the endogenous debt capacity  $|\underline{S}_t|$  is proportional to  $K_t$ . By using the homogeneity property, we write  $s_t = S_t/K_t$  and correspondingly  $\underline{s} = \underline{S}_t/K_t$ . The constraint (73) can be written as  $\underline{s} < s_t < 0$  where  $\underline{s} < 0$  is a time-invariant constant to be determined as part of our model solution (via boundary conditions.)

As for the one-sided commitment case, the volatility of scaled savings  $s$  has to approach to zero at both boundaries  $\underline{s}$  and the origin. That is, in addition to (44)-(45), we also have

$$\lim_{s \rightarrow 0} \sigma_s(s) = \lim_{s \rightarrow 0} \frac{\sigma \gamma h(s) h'(s)}{h(s) h''(s) - \gamma h'(s)^2} = 0, \quad (74)$$

which implies

$$\lim_{s \rightarrow 0} h''(s) = -\infty. \quad (75)$$

In summary, in the implementation formulation, the entrepreneur's scaled certainty equivalent wealth  $h(s)$  solves ODE (43) subject to the boundary conditions (44)-(45) and (75).

## 7.2 Results

In this section, we compare the results for the general case under two-sided limited commitments with those for the case with one-sided limited commitment. We use the same parameter values as in Section 6.

Figure 5 shows that value losses for the two-sided case are much greater. For example, in our baseline case, the value loss  $q^{FB} - p(w)$  ranges from 0 to 0.027. At the endogenous upper boundary  $\bar{w} = 0.984$  or equivalently  $s = 0$ , the value loss  $q^{FB} - p(\bar{w}) = q^{FB} - e(0) = 0.216$ , which is about 18% percent of the first-best value  $q^{FB} = 1.20$ . Second, the marginal value of

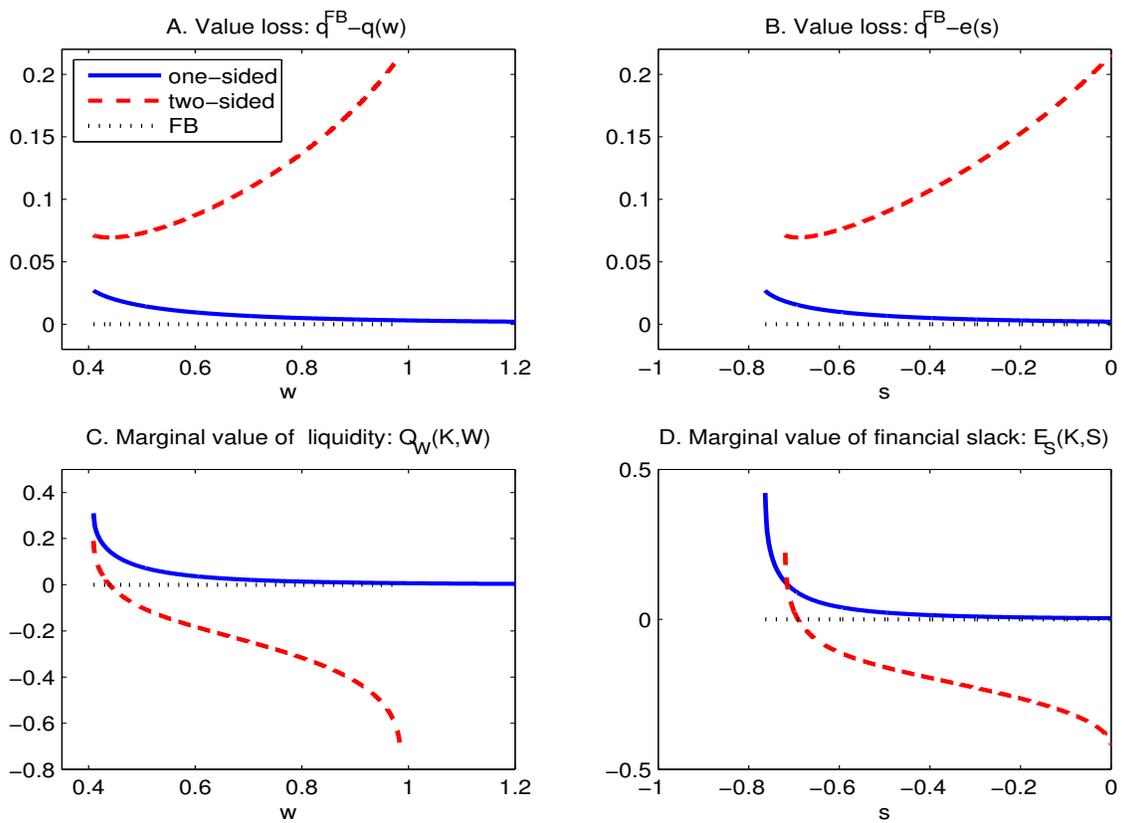


Figure 5: Value losses and the marginal values of liquidity/financial slack.

liquidity  $P_W(K, W) = p'(w)$  or equivalently  $E_S(K, S) = e'(s)$  can be negative with two-sided limited commitments. The intuition is as follows. With sufficiently high  $w$  or equivalently  $s$ , investors' limited liability becomes more likely to bind. In order to mitigate the investors' limited liability constraint, the firm wants to reduce the scaled liquidity  $w$ . That is, a high  $w$  can be in net costly for the firm as a while.

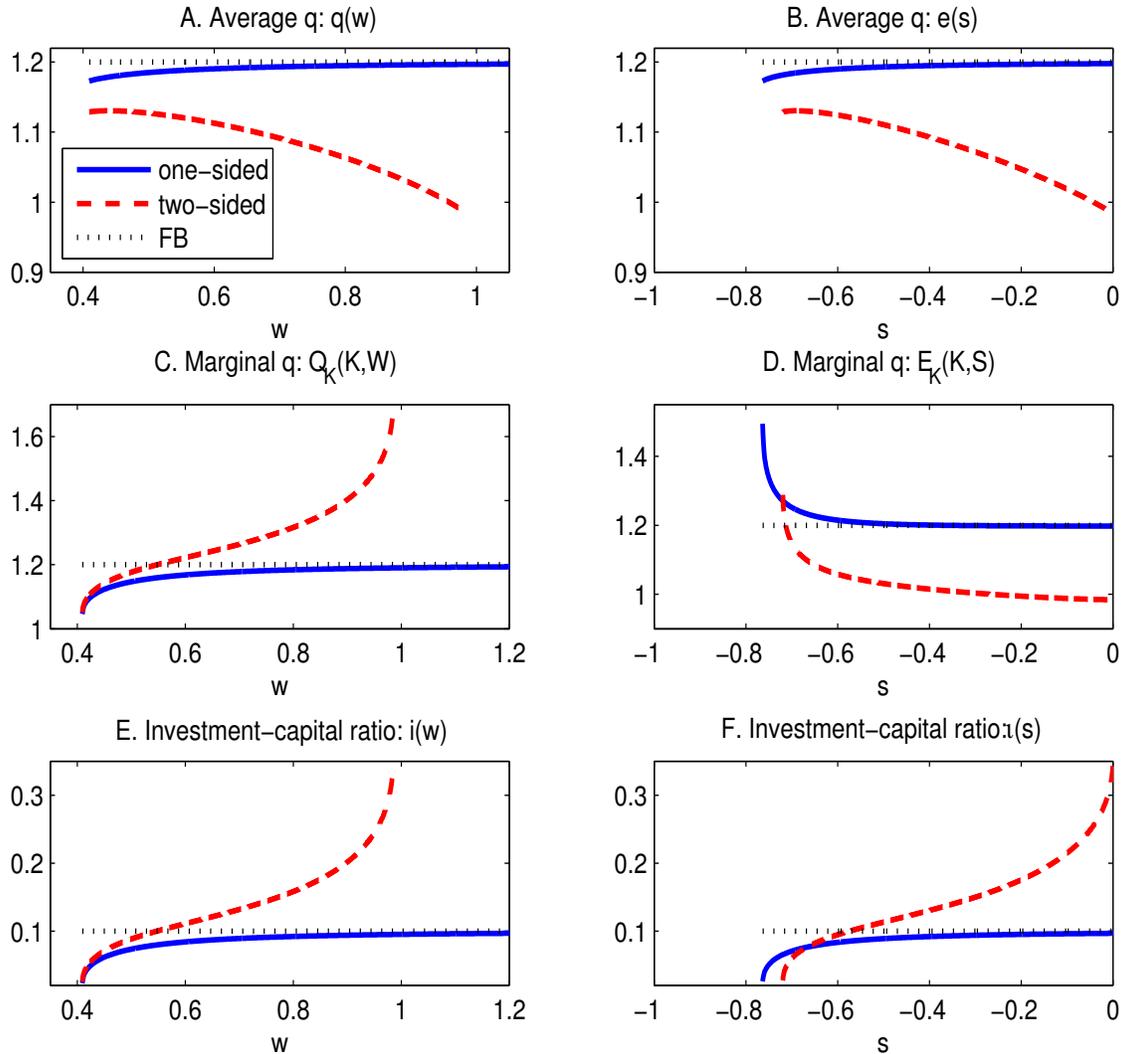


Figure 6: **Average  $q$ , marginal  $q$  and the investment-capital ratio  $i$  for both of optimal contracting and implementation formulations.**

Figure 6 plots average  $q$ , marginal  $q$ , and the investment-capital ratio for both one-sided and two-sided limited commitment cases. First, the additional limited liability constraint

for investors substantially lowers the firm’s enterprise value (or Tobin’s average  $q$ ) as we see in Panels A and B of Figure 6. Second, marginal  $q$  measured by  $P_K$  in the contracting formulation for the two-sided case can be much higher especially for high values of  $w$  and correspondingly, investment-capital ratio  $i(w)$  is very high reaching more than 30% at  $\bar{w} = 0.984$ , as we see from Panels C and E, which clearly indicate the firm’s incentives to *over*-invest. The intuition is as follows. By significantly increasing its investment  $I$ , the firm will be much bigger in expectation and hence lowering  $w = W/K$ , *ceteris paribus*. In the implementation formulation, while marginal  $q$ , measured by  $E_K$  is lower for most values of  $s$ , the firm’s marginal value of liquidity is also much lower. As Panel F shows, the net effect still yields excessive investment from the social perspective. In summary, we clearly

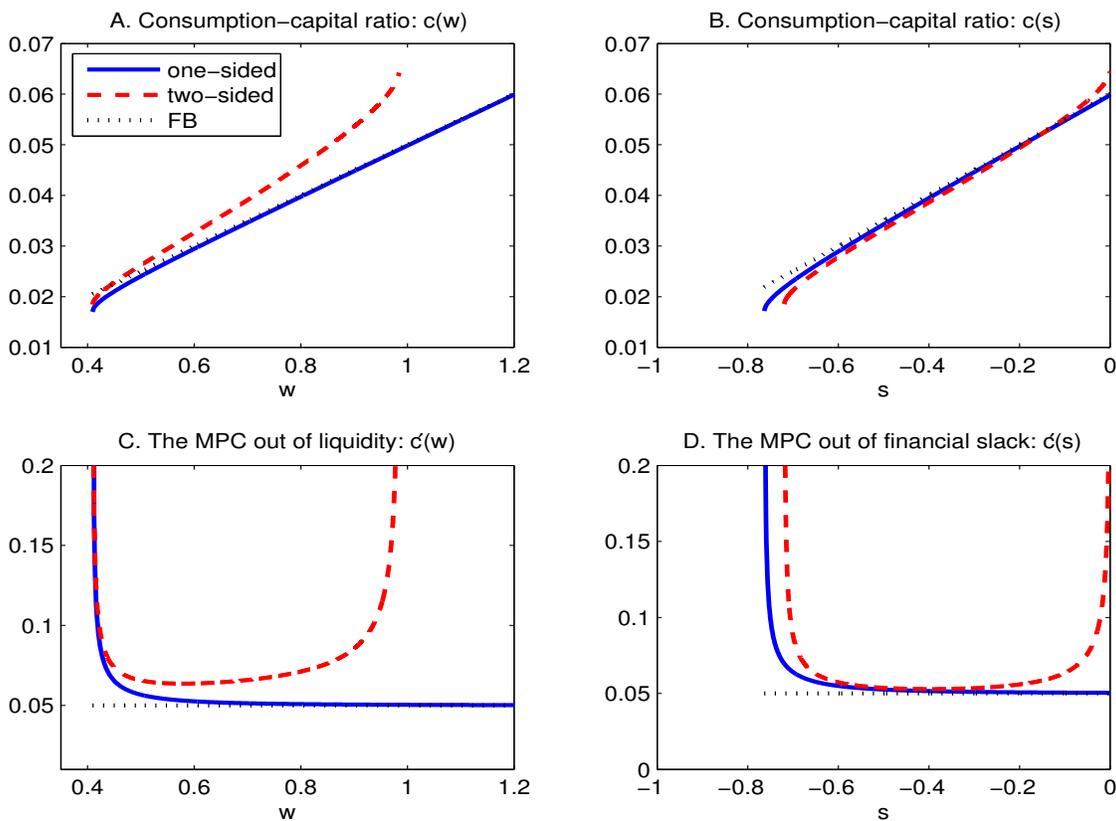


Figure 7: **Optimal consumption policies and the MPCs.**

that introducing limited liability constraint for investors substantially change the economics. Rather than under-vest, the firm may over-invest. Importantly, over-investment here is not induced by the firm’s incentive to take excessive risk at creditors’ expense. Instead, the

firm wants to optimally manage the entrepreneur's promised wealth  $w$  or equivalently the entrepreneur's scaled savings  $s$  within an interior region.

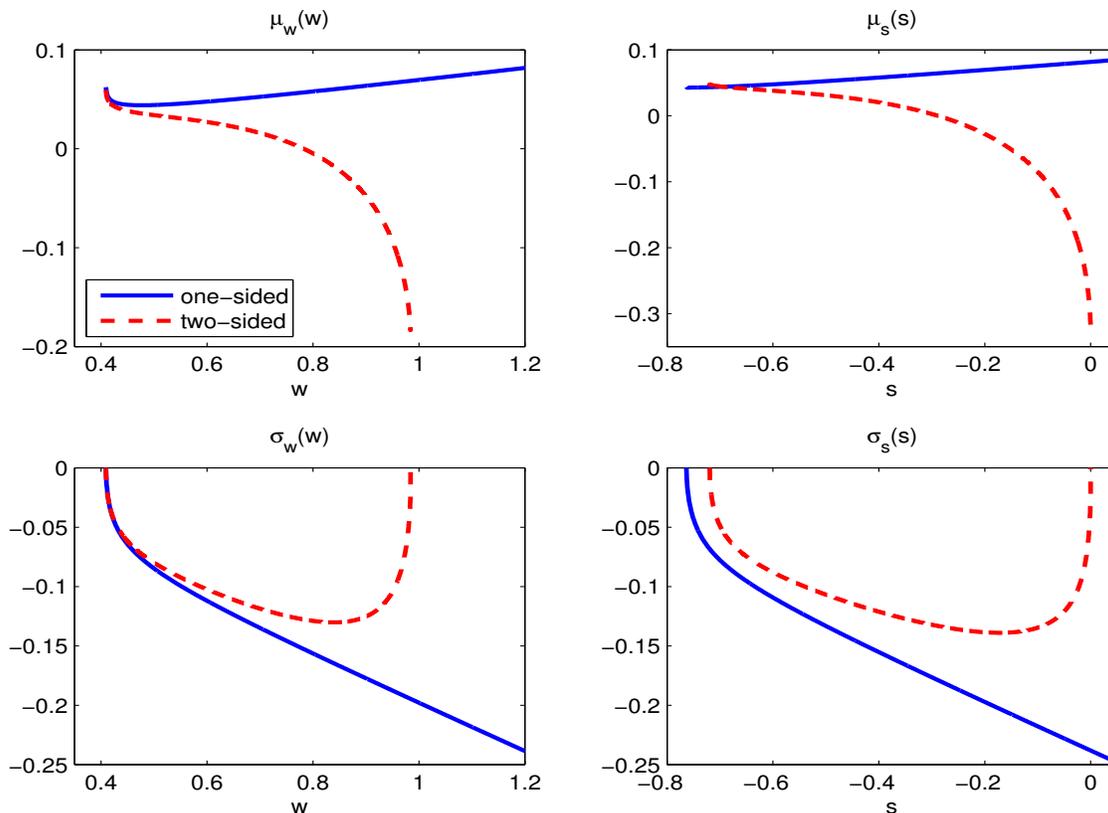


Figure 8: **The drift and volatility functions for promised wealth  $w$  and savings  $s$ .** Note that  $\sigma_w(\hat{q}) = 0$  for both cases. For the case with two-sided limited commitments, in both formulations, at the corresponding upper boundary, we have  $\sigma_w(\bar{w}) = 0$  and  $\sigma_s(0) = 0$ .

Next, we turn to the entrepreneur's consumption. Importantly, because investors have to satisfy their limited liability constraints at all times, the entrepreneur now has incentives to consume more even than the first-best consumption level (see Panels A and B of Figure 7). The intuition is as follows. By consuming more than the first-best level, the entrepreneur's promised wealth can be lowered (as the entrepreneur is consuming more now). As a result, it relaxes the investors' limited liability constraint (going forward), which is in both parties' interests. As a result, consumption becomes much more nonlinear. Indeed, now consumption is concave in liquidity from low to medium levels of  $w$  and equivalently  $s$ , but is convex in  $w$  and  $s$  from medium to high levels of values. As a result, the MPCs are no longer monotonic

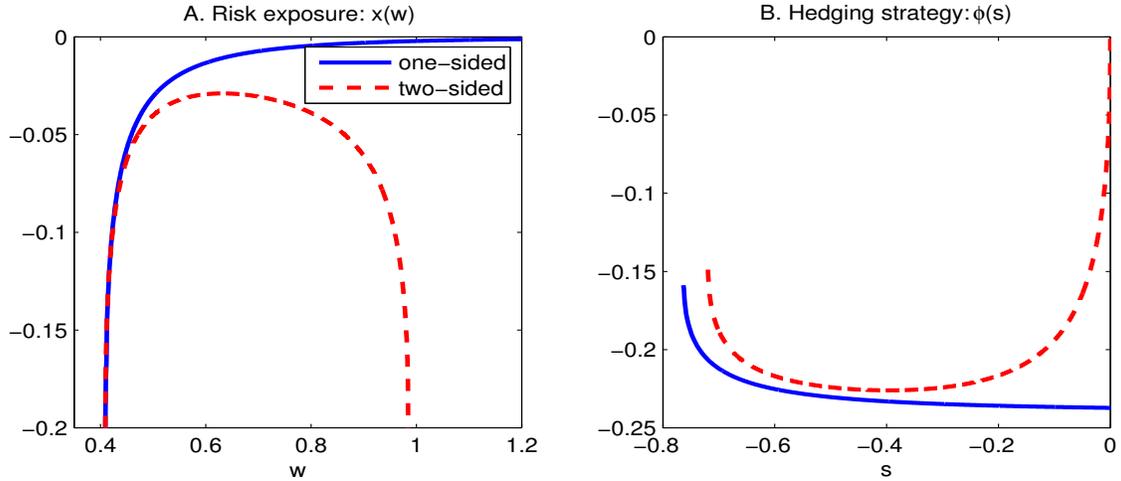


Figure 9: **Optimal risk exposure  $x(w)$  and hedging  $\psi(s)$ .**

as we see from Panels C and D of Figure 7.

Figure 8 plots the drift and the volatility functions for the scaled  $w$  and scaled savings  $s$ . Note that at the boundaries, volatilities for both  $w$  and  $s$  approach zero in order to ensure that the boundaries are not crossed. Moreover, at the lower boundary, we need to verify that the drift is positive so that  $s$  and  $w$  moves towards the interior of the region. Indeed, at the lower boundary  $\hat{q} = 0.41$  and  $\underline{s} = -0.764$ ,  $\mu_w(\hat{q}) > 0$  and  $\mu_s(\underline{s}) > 0$ , as we see from Panels A and B of Figure 8, respectively. For the two-sided limited commitments case, at the upper boundary  $\bar{w} = 0.984$  and  $s = 0$ , we have  $\mu_w(\bar{w}) > 0$  and  $\mu_s(0) > 0$ , which ensure that the scaled promised wealth  $w$  and the scaled savings  $s$  will drift towards the interior region.

Figure 9 plots the risk exposure  $x(w)$  in the contracting formulation and hedging strategy  $\psi(s)$  in the implementation formulation. In Panel A, with one-sided commitment, self insurance is sufficient as  $w \rightarrow \infty$ , and hence  $x(w) \rightarrow 0$  as  $w \rightarrow \infty$ . With two-sided commitments, recall that  $\lim_{w \rightarrow \bar{w}} g''(w) = -\infty$  and notice risk exposure given in (36), we have  $x(\bar{w}) = (1 - \gamma)\sigma = -0.2$  as shown in panel A of Figure 9. In Panel B, with one-sided commitment, the hedging strategy is constant and equates to  $\lim_{s \rightarrow \infty} \psi(s) = -\sigma q^{FB} = -0.24$ . With two-sided commitments, noticing (49) the condition  $\lim_{s \rightarrow \underline{s}} h''(s) = -\infty$  implies the hedging strategy at the boundary  $s = 0$  is  $\lim_{s \rightarrow \infty} \psi(s) = \sigma s = 0$ . Figure 9 shows that the principal engages in substantial asset sales and do not engage in risk management when the firm faces high financial distress, especially at the right boundary  $w = \bar{w}$  or equivalently

$s = 0$  under two-sided commitments case.

### 7.3 Simulation

We now use our model solution to simulate a sample path and highlight the economic implications of our model. Figures 10 and 11 report the simulation results. On the left columns of both figures, we report the simulated path for the optimal contracting problem. On the right columns of both Figures 10 and 11, we plot the simulated path for the financial implementation problem. All plots are based on one simulated Brownian shock  $Z$  path.

In Figure 10, Panels A and B plot dynamics of capital  $K$  for both the contracting problem and the financial implementation. Interestingly, capital  $K$  turns out to be higher for the two-sided commitment case due to over-investment incentives. As the firm always under-invests in the one-sided case, the capital process for the first-best case always lies on top of the  $K$  process under the one-sided commitment. Also, as we expect, for all the three cases, the first-best, the one-sided commitment and the two-sided commitments, the simulated sample paths in the two formulations match perfectly as these two optimization problems are economically equivalent. Panels C and E plot the simulated paths for promised certainty equivalent wealth  $W$  and consumption  $C$ . Under the first-best,  $W$  is flat as consumption is fully insurance and flat over time. For the one-sided commitment case, consumption is non-decreasing, remains flat for a period of time, and adjusts upward only when the limited enforcement constraint binds. For the two-sided commitment case, we see that consumption can either increase or decrease. When neither the investors' limited liability nor the entrepreneur's limited commitment constraint binds, consumption will remain unchanged from the last period and hence flat for a region. Again, note that the consumption process  $C$  implied by the two specifications are identical as we see from Panels E and F.

Now we turn to Figure 11. Panels A and B report the sample paths for consumption-capital ratio  $c$  for the three cases. Note that while consumption in levels  $C$  can be flat over a period of time, consumption-capital ratio  $c_t$  is stochastic. For this simulated capital shock  $Z$  path, eventually  $c$  is highest for the one-sided commitment case because of under-investment and hence  $K$  is lower which helps to increase  $c = C/K$ .

In terms of investment  $I$  in levels, there is no strict ranking across the three cases. For

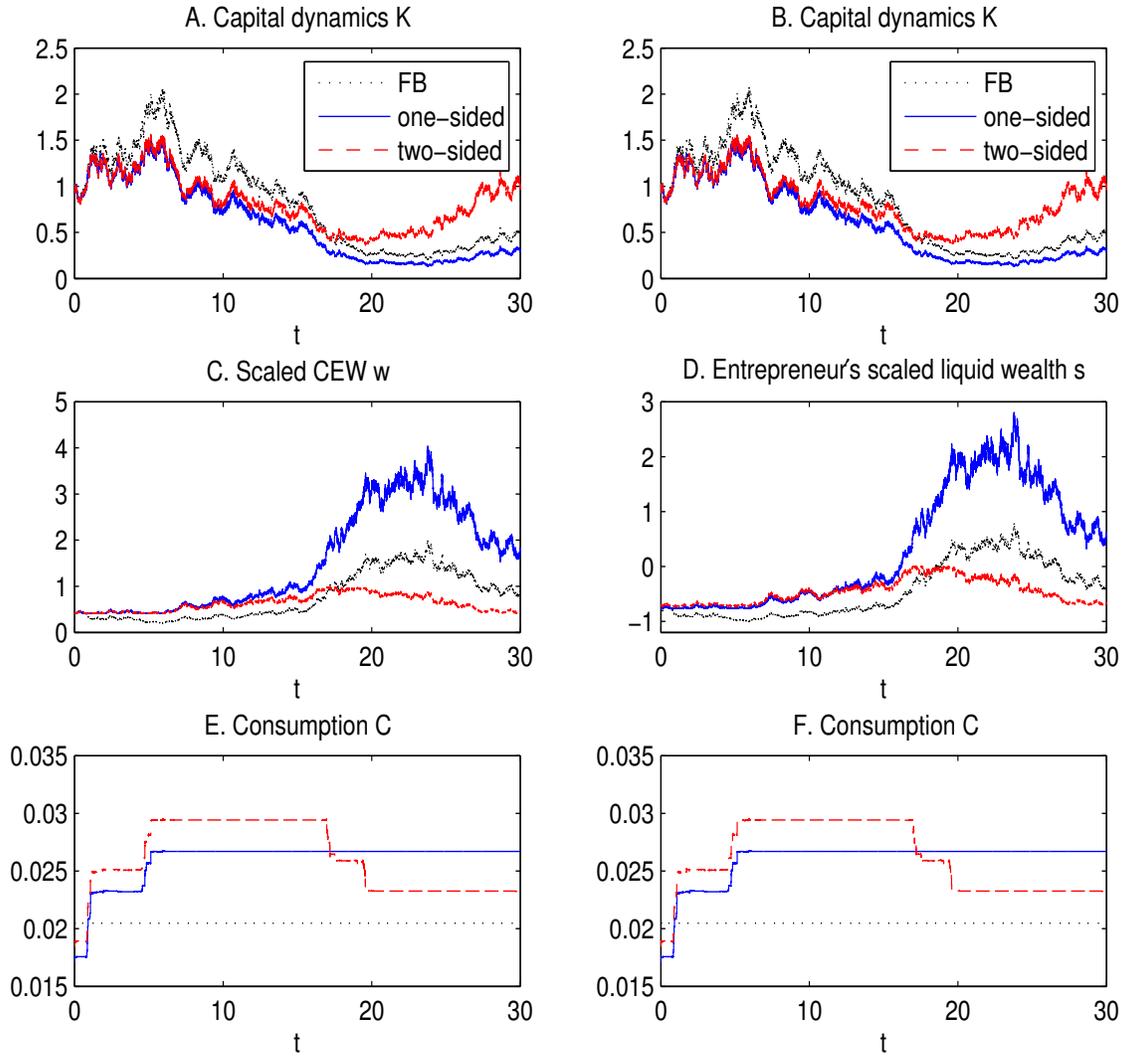


Figure 10: **Simulated sample paths for the cases with one-sided and two-sided limited commitments.** The left column reports the simulation results for the contracting problem and the right column reports the results for the financial implementation. Note that the simulated paths are identical for all common economic variables in the two formulations such as  $K$  and  $C$ .

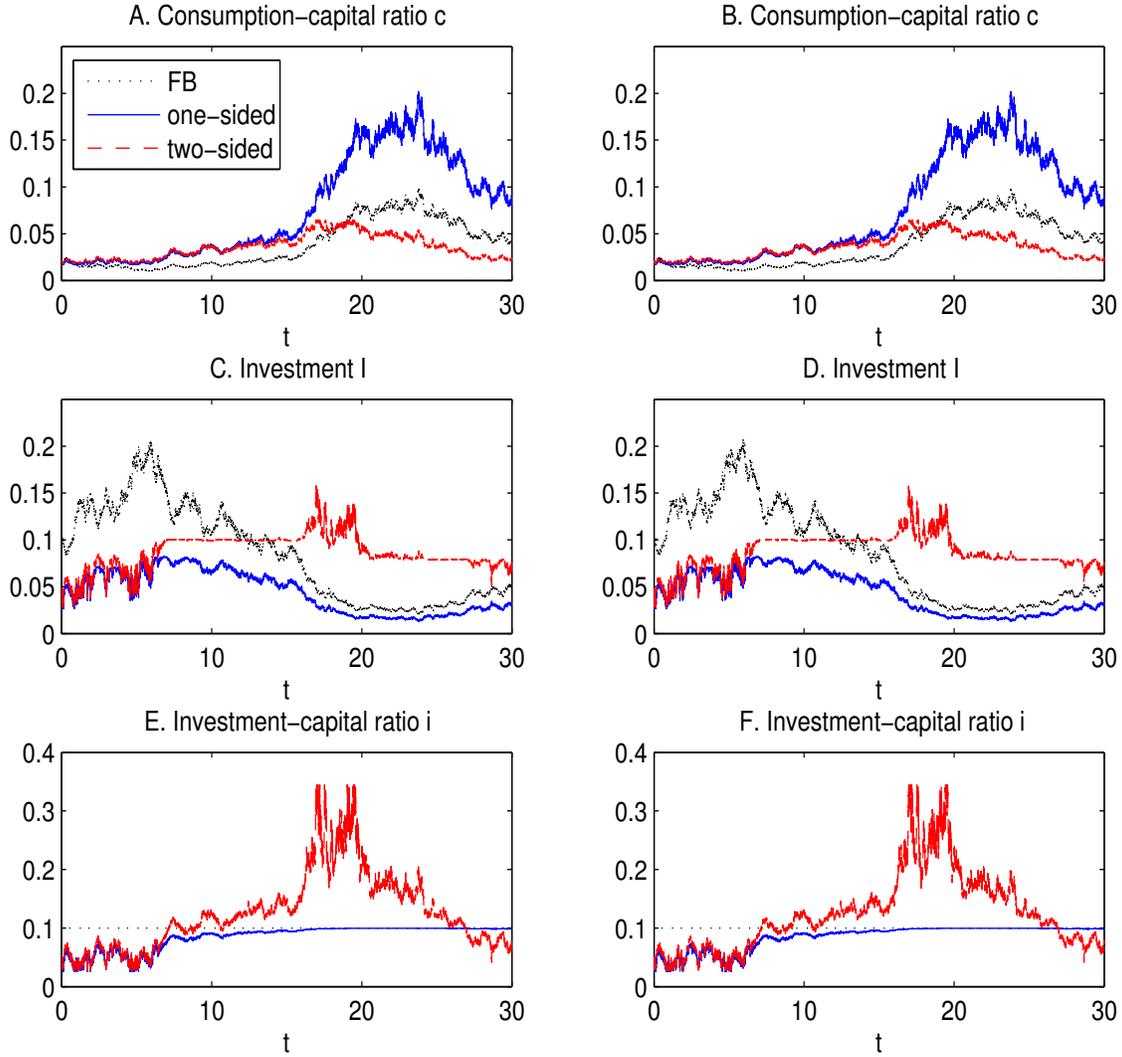


Figure 11: **Simulated sample paths for the cases with one-sided and two-sided limited commitments.** The left column reports the simulation results for the contracting problem and the right column reports the results for the financial implementation. Note that the simulated paths are identical for all common economic variables in the two formulations such as  $c$  and  $I$  and  $i$ .

this path, investment  $I$  for the two-sided limited commitment case eventually is the highest due to the firm's incentive to over-invest. If we analyze the investment-capital ratio  $i$ , the degree of over-investment is more striking. Note that under one-sided commitment, the firm eventually hoards sufficient slack and effectively achieves the first-best investment level  $i^{FB} = 10\%$  in year 15.

## 8 Risk and returns

In this section, we analyze the asset pricing implications of our contracting problem. We decompose the original shock to capital  $Z$  into two components,  $B^m$  and  $B^u$ . That is, we rewrite the dynamics for capital accumulation (1) as follows,

$$dK_t = (I_t - \delta K_t)dt + \sigma K_t(\rho dB_t^m + \sqrt{1 - \rho^2}dB_t^u). \quad (76)$$

where we assume that the two standard Brownian motions  $B^m$  and  $B^u$  are uncorrelated. As we will make it clear below,  $B^m$  is the systematic shock and  $B^u$  is the idiosyncratic component.

Specifically, we introduce two new risky assets which can be continuously traded without frictions in the public market. Assume that the incremental return  $dR_t^m$  of the first risky asset, which we refer to as the market portfolio, over time period  $dt$  is i.i.d.,

$$dR_t^m = \mu_m dt + \sigma_m dB_t^m, \quad (77)$$

where  $\mu_m$  and  $\sigma_m$  are constant mean and volatility parameters of this return process. Let

$$\eta = \frac{\mu_m - r}{\sigma_m} \quad (78)$$

denotes the Sharpe ratio of the market portfolio.

We assume that the incremental return  $dR_t^u$  of the second risky asset over time period

$dt$  is also i.i.d., but with no risk premium, in that

$$dR_t^u = rdt + \sigma_u dB_t^u, \quad (79)$$

where  $\sigma_u$  is constant volatility parameters of the second asset process.

By construction, now the shocks to the capital stock  $K$  is fully spanned. Without contracting frictions, the market is complete. However, limited enforcements of the long-term contracts will still distort investment and other corporate decisions. We will derive conditional asset pricing and provide new financial implementations. Intuitively, we may interpret  $B^u$  as the Brownian motion that drives the idiosyncratic risk component of the capital shock and hence carries no risk premium for fully diversified investors.

## 8.1 Solution

For the case with two-sided full commitments, MM theorem holds and we achieve the first-best outcome. Additionally, the standard CAPM holds by construction.

**Proposition 4** *Under full commitment, Tobin's average  $q$  is given by*

$$q^{FB} = \max_i \frac{A - i - \phi(i)}{r + \delta + \rho\eta\sigma - i}. \quad (80)$$

*and the firm's first-best investment-capital ratio  $i^{FB}$  is the maximand of (80). The capital asset pricing model(CAPM) holds for the firm with its expected return given by*

$$r + \rho\eta\sigma = r + \beta^{FB}(\mu_m - r), \quad (81)$$

*where the firms beta,  $\beta^{FB}$ , is constant and given by*

$$\beta^{FB} = \frac{\rho\sigma}{\sigma_m}. \quad (82)$$

Comparing with the first-best Tobin's  $q$  and investment-capital ratio  $i$  in Proposition 1 for the case with risk-neutral investor, we note that  $\rho\eta\sigma$  in the denominator of (80) reflects the systematic risk premium as captured by CAPM.

We now summarize the main results for the case with two-sided limited commitments.

**Proposition 5** *With two-sided limited commitments, the investors' scaled value function  $g(w)$  solves the following ODE,*

$$\begin{aligned}
rg(w) = & A + \frac{m\gamma}{1-\gamma} (-g'(w))^{1/\gamma} w + i(w)(g(w) - wg'(w) - 1) - \phi(i(w)) + \frac{\zeta}{1-\gamma} wg'(w) \\
& - (\delta + \rho\eta\sigma)(g(w) - wg'(w)) + \frac{\sigma^2 w^2}{2} \frac{\gamma g'(w)g''(w)}{wg''(w) + \gamma g'(w)} - \frac{\eta^2}{2} \frac{wg'(w)^2}{wg''(w) + \gamma g'(w)} \\
& - \rho\eta\sigma \frac{w^2 g'(w)g''(w)}{wg''(w) + \gamma g'(w)}
\end{aligned} \tag{83}$$

subject to the following boundary conditions

$$g(\bar{w}) = 0, \tag{84}$$

$$\lim_{w \rightarrow \bar{w}} \sigma_w(w) = \lim_{w \rightarrow \bar{w}} \frac{\sigma \gamma w g'(w)}{wg''(w) + \gamma g'(w)} = 0, \tag{85}$$

$$\lim_{w \rightarrow \hat{q}} \sigma_w(w) = \lim_{w \rightarrow \hat{q}} \frac{\sigma \gamma w g'(w)}{wg''(w) + \gamma g'(w)} = 0, \tag{86}$$

where

$$m = \zeta + (1 - \gamma^{-1}) \left( r - \zeta + \frac{\eta^2}{2\gamma} \right). \tag{87}$$

The investment-capital ratio  $i = I/K$ , the consumption-capital ratio  $c = C/K$ , the systematic risk exposure  $x^m$  and the idiosyncratic risk exposure  $x^u$  are given by

$$\phi'(i(w)) = g(w) - wg'(w) - 1, \tag{88}$$

$$c(w) = m (-g'(w))^{1/\gamma} w, \tag{89}$$

$$x^m(w) = \frac{(1-\gamma)\eta g'(w)}{wg''(w) + \gamma g'(w)} + \frac{(1-\gamma)\rho\sigma w g''(w)}{wg''(w) + \gamma g'(w)}, \tag{90}$$

$$x^u(w) = \frac{(1-\gamma)\sqrt{1-\rho^2}\sigma w g''(w)}{wg''(w) + \gamma g'(w)}. \tag{91}$$

We can again provide a financial implementation. The following proposition summarizes the main results for the implementation of the optimal contract for the case with two-sided limited commitments.

**Proposition 6** *Under two-sided limited commitments, the scaled certainty equivalent value  $h(s)$  solves the following ODE,*

$$0 = \frac{h(s)}{1-\gamma} \left[ \gamma m h'(s)^{\frac{\gamma-1}{\gamma}} - \zeta \right] - \delta h(s) + [(r+\delta)s + A] h'(s) + \iota(s)(h(s) - (s+1)h'(s)) - \phi(\iota(s))h'(s) \\ - \frac{\gamma\sigma^2}{2} \frac{h(s)^2 h''(s)}{h(s)h''(s) - \gamma h'(s)^2} - \frac{\eta^2}{2} \frac{h(s)h'(s)^2}{h(s)h''(s) - \gamma h'(s)^2} - \rho\eta\sigma h'(s) \frac{\gamma h'(s)(h(s) - sh'(s)) + sh(s)h''(s)}{\gamma h'(s)^2 - h(s)h''(s)},$$

subject to the following boundary conditions:

$$h(\underline{s}) = \hat{q}, \quad (92)$$

$$\lim_{s \rightarrow \underline{s}} \sigma_s(s) = \lim_{s \rightarrow \underline{s}} \frac{\sigma\gamma h(s)h'(s)}{h(s)h''(s) - \gamma h'(s)^2} = 0, \quad (93)$$

$$\lim_{s \rightarrow 0} \sigma_s(s) = \lim_{s \rightarrow 0} \frac{\sigma\gamma h(s)h'(s)}{h(s)h''(s) - \gamma h'(s)^2} = 0, \quad (94)$$

where  $m$  is given by (87). The consumption-capital ratio  $c(s)$ , the investment-capital ratio  $\iota(s) = I/K$ , and the hedging policies  $\psi^m(s)$  and  $\psi^u(s)$  are given by

$$c(s) = mh(s)h'(s)^{-1/\gamma}, \quad (95)$$

$$\phi'(\iota(s)) = \frac{h(s)}{h'(s)} - s - 1, \quad (96)$$

$$\psi^m(s) = \frac{\eta h(s)h'(s)}{\gamma h'(s)^2 - h(s)h''(s)} + \rho\sigma \frac{\gamma h'(s)(h(s) - sh'(s)) + sh(s)h''(s)}{h(s)h''(s) - \gamma h'(s)^2}, \quad (97)$$

$$\psi^u(s) = \sqrt{1 - \rho^2\sigma} \frac{\gamma h'(s)(h(s) - sh'(s)) + sh(s)h''(s)}{h(s)h''(s) - \gamma h'(s)^2}. \quad (98)$$

Figure 12 plots the systematic risk exposure  $x^m(w)$  and idiosyncratic risk exposure  $x^u(w)$  in the contracting formulation and hedging strategy  $\psi^m(s)$  and  $\psi^u(s)$  in the implementation formulation. In Panel A, with one-sided commitment, the systematic risk exposures at the two boundaries are  $x^m(\hat{q}) = (1-\gamma)\rho\sigma = 0.1$  and  $\lim_{w \rightarrow \infty} x^m(w) = (1-\gamma)\eta/\gamma = -0.15$ , respectively. With two-sided commitments, the systematic risk exposures at the two boundaries are equal and given by  $x^m(\hat{q}) = x^m(\bar{w}) = (1-\gamma)\rho\sigma = 0.1$ . In Panel B, with one-sided commitment, the idiosyncratic risk exposure is  $x^u(\hat{q}) = (1-\gamma)\sqrt{1-\rho^2}\sigma = -0.173$  and  $\lim_{w \rightarrow \infty} x^u(w) = 0$ . With two-sided commitments, we have the idiosyncratic risk exposure is  $x^u(\hat{q}) = x^u(\bar{w}) = (1-\gamma)\sqrt{1-\rho^2}\sigma = -0.173$ .

In Panel C, with one-sided commitment, the hedging strategy is  $\psi^m(\hat{q}) = \rho\sigma\hat{q}$  and  $\lim_{s \rightarrow \infty} \psi^m(s) = \frac{\eta}{\gamma}h(s) - \rho\sigma q^{FB}$ . With two-sided commitments, the hedging strategy is  $\psi^m(\hat{q}) = \rho\sigma\hat{q}$  and  $\psi^m(0) = \rho\sigma 0 = 0$ . In Panel D, with one-sided commitment, the hedging strategy is  $\psi^u(\underline{s}) = \sqrt{1 - \rho^2}\sigma\underline{s}$  and  $\lim_{s \rightarrow \infty} \psi^u(s) = -\sqrt{1 - \rho^2}\sigma q^{FB}$ . With two-sided commitments, the hedging strategy is  $\psi^u(\underline{s}) = \sqrt{1 - \rho^2}\sigma\underline{s}$  and  $\psi^u(0) = \sqrt{1 - \rho^2}\sigma \times 0 = 0$ . In summary, we see that there is significant non-linearity in risk exposure and hedging under both one-sided and two-sided commitments. Moreover, whether the commitments are one-sided or two-sided make significant differences on corporate risk management.

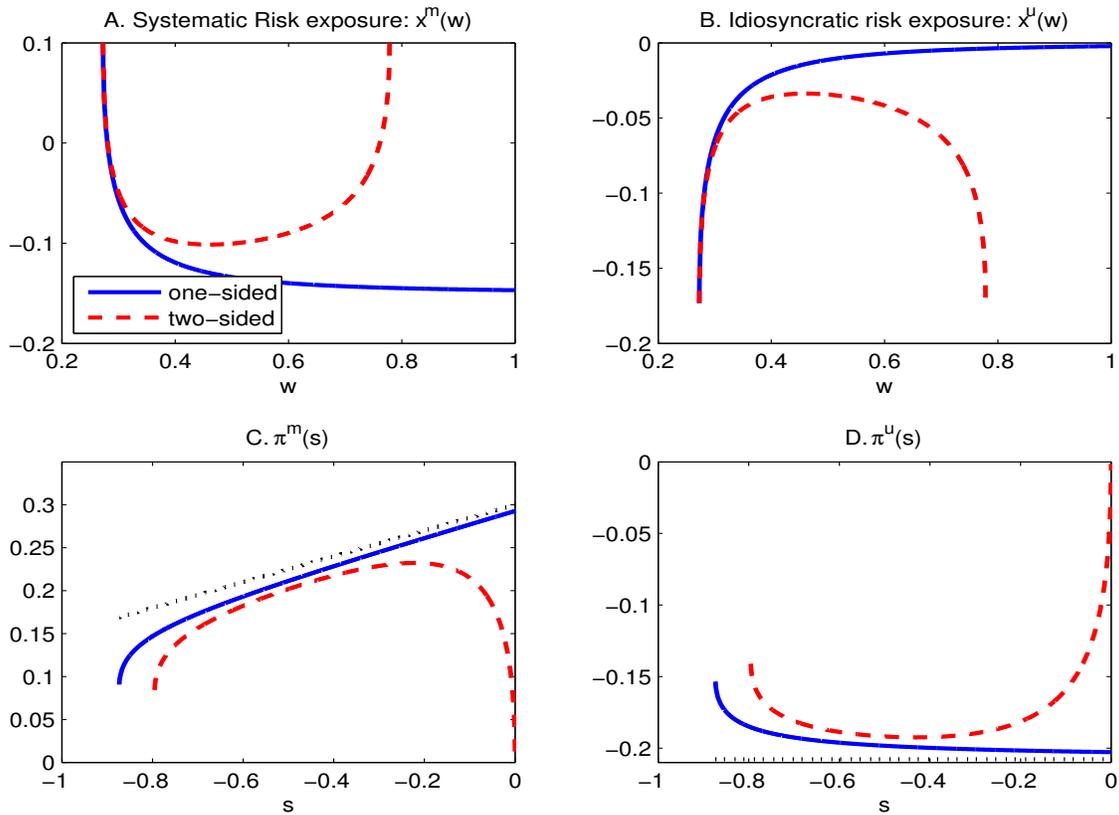


Figure 12: **Optimal risk exposure and hedging strategies.** Parameter values:  $\rho = -0.5$ ,  $\eta = 0.3$ , and  $\hat{q} = 0.27$ .

## 9 Conclusion

We analyze a dynamic optimal contracting model where the entrepreneur has risky inalienable human capital and investors have limited liability protection. We show that corporate risk management and liquidity management are critical components of optimal corporate financial policies for firms operating in economic environments with limited contractual enforcements. We derive endogenous debt capacity that ties to the entrepreneur's risky inalienable human capital and the investors' limited liability constraints. We show that corporate investment and the entrepreneur's consumption dynamics are highly nonlinear especially when the firm's liquidity is low. Investment distortions via asset sales are critical parts of corporate risk management in bad times. Preserving liquidity and financial flexibility is of the first-order importance for corporate decision making. Firms facing high financial distress engage in substantial asset sales and do not engage in risk management. Additionally, over-investment can be highly valuable for firms to mitigate their concerns to violate limited liability conditions. However, the fact that we observe firms' excessive investment near bankruptcy is no indication that equity holders are engaging in excessive risk taking. Finally, we quantitatively value the net benefits of risk management and liquidity management. Finally, we find that the quantitative implications of limited liability protection for investors are substantial. For example, in our baseline calibration, the firm's enterprise value is about 20% lower when the firm's limited liability constraint is about to bind.

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# Appendices

## A Technical details

To be added.