NEGATIVE BUBBLES UNDER SHORT-SALES CONSTRAINTS AND HETEROGENEOUS BELIEFS*

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Abstract

In a dynamic trading model, investors with heterogeneous beliefs have an option to sell the stock now and buy it back later. Due to this repurchase option and investors' risk aversion, the stock price might be lower than the lowest valuation among investors even when the short-sales constraint is binding. This result compliments that of Harrison and Kreps (1978), in which due to a resale option and risk neutrality, a (positive) bubble always arises. We also demonstrate that in a static model, neither a bubble nor a negative bubble exists.

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1 Introduction

In a seminal paper, Harrison and Kreps (1978) analyze how heterogeneous beliefs affect stock prices[1] with short-sales constraints and risk-neutral investors. If investors were to hold the stock forever after purchasing it, the most optimistic investor would purchase all of the stock, and the stock price would be equal to the highest valuation among investors.[2] With dynamic trading, however, an investor can buy the stock now and sell it to the investor who has the highest conditional valuation in the future. They term the ability to sell in the future a resale option. An investor may have the highest valuation today on an expected basis, but she may not have the highest valuation in all future states. Consequently, the equilibrium stock price under heterogeneous beliefs is always higher than the highest valuation among investors. According to their definition, this resale option gives rise to a (positive) bubble.[3]

Because of the short-sales constraint and investors' risk neutrality in Harrison and Kreps, the stock price in every state of nature is determined by the investor who has the highest valuation in that state, because the short-sales constraint binds for all but the most optimistic investor in the state. Pessimistic investors have no effect on the stock price even when they sell all of their holdings, because risk-neutral investors do not demand a risk premium for holding risky stocks. The stock price will never be lower than the valuation of the most pessimistic investor. Harrison and Kreps claim that the risk neutrality of investors is only for simplicity for the existence of bubbles.

In this paper, we show that the risk neutrality is crucial for the existence of bubbles. Specifically, we demonstrate that with short-sales constraints and heterogeneous beliefs about the stock payoffs, negative bubbles, which are defined as the stock price being lower than the valuation of the most pessimistic investor, can arise with any strictly risk-averse investors, even if the short-sales constraint is binding. The key intuition of our result is that although pessimistic investors cannot sell short, which limits their impact on the stock price, they can always sell the stock that they already own, putting the price pressure on the stock

[1] A short list of models of heterogeneous beliefs includes Varian (1989), Harris and Raviv (1993), Kandel and Pearson (1995), Morris (1996), Hong and Stein (2003), Scheinkman and Xiong (2003), Allen, Morris, and Shin (2006), Banerjee, Kaniel, and Kremer (2009), Cao and Ou-Yang (2009), Cao, Ou-Yang, and Ye (2016), and Dumas, Lewis, and Osambela (2017). Scheinkman and Xiong (2004) is a review of this literature.

[2] This scenario occurs because all investors are assumed to be wealthy enough to buy up all of the stock, trade competitively in the stock market, and face short-sales constraints.

[3] See Brunnermeier and Oehmke (2013) for a survey of literature on bubbles.

because optimistic investors are risk averse and demand a premium for bearing more risk.[4] With short-sales constraints and risk aversion, dynamic trading has two competing effects on the stock price: the expectation effect and the risk-premium effect. On one hand, the short-sales constraint prevents pessimistic investors from participating in the stock market, and hence the constraint tends to increase the expectation of the final stock payoffs. On the other hand, risk-averse optimistic investors have to hold a larger position in the stock or even the entire stock supply, and thus they demand a larger risk premium. When the risk-premium effect dominates the expectation effect, the stock price can be lower than the lowest valuation among investors, resulting in a negative bubble.

Generally, we show that with strictly risk-averse investors and heterogeneous beliefs, there always exist parameter values under which dynamic trading leads to negative bubbles. A key insight is that when some investors expect that with a high probability, some other investors in the market will become pessimistic in the future, they can sell the stock now and hope to buy it back later at an even lower price. We term this investor behavior the repurchase option, as opposed to the resale option as in Harrison and Kreps. A resale option is that investors buy overvalued stock at a low price today because they believe that other investors will become even more optimistic in the future, they hope to sell the stock to those optimistic investors. A repurchase option is that relatively optimistic investors sell the stock today because they believe that other investors will become very pessimistic in the future, they can then buy back the stock from those pessimistic investors at an even lower price. Due to risk neutrality and short-sales constraints in Harrison and Kreps, the repurchase option does not exist, because the selling of pessimistic investors does not affect the stock price. When investors are risk averse, however, the selling of pessimistic investors matters and the repurchase option becomes relevant.

Interestingly, in a one-period model with heterogeneous beliefs, such as Miller (1977), we find that the short-sales constraint always increases the stock price. One can decompose the stock price into two components: the expectation of the stock payoff and the risk premium. The risk premium is proportional to both the number of shares of the stock that marginal investors hold and the stock volatility. In a model of heterogeneous beliefs, no learning occurs, and hence the short-sales constraint does not change the beliefs of investors. The

[4]Under the assumption of risk neutrality, the selling from the pessimistic investors has no effect on the stock price, and hence the stock price is always higher with dynamic trading.

constraint prevents the pessimistic investors from trading in the market, and thus stock prices tend to reflect a more optimistic valuation than they otherwise would. Therefore, the constraint increases the expectation component in the stock price and also reduces the number of shares of the stock that the optimistic investors hold, which decreases the risk-premium component. Consequently, the short-sales constraint always increases the stock price. We further show, however, that in all of the static or one-period models, neither bubbles nor negative bubbles arise, due to the absence of resale options and repurchase options. In other words, the equilibrium stock price in a static model cannot be higher than the highest valuation or lower than the lowest valuation among investors.

A potential implication of our model is that when investors have divergent views about the stock payoffs, even with the short-sales constraint, more trading may still reduce the stock price. In the summer of 2015, the Shanghai stock index in China dropped 43% from 5166 to 2927 in two months. Short selling was not illegal but not practical, as the major brokerage firms voluntarily ceased to provide short-selling services and the government imposed heavy limits on index futures trading. During this market decline, one third of the publicly traded firms applied for trading halts for an indefinite period of time. After the market stabilized, the stocks subject to trading halts renewed trading by submitting a plea to the China Securities Regulatory Commission. One potential explanation for those companies to apply for trading halts is that even optimistic investors, who are supposed to buy the stock today, may choose to wait or even sell, hoping to buy the stock at an even lower price later, when other pessimistic investors sell. The opportunity to trade in the future would drive down the stock price today. Sharp price drops could create problems for these companies because many of them had used their stocks as collateral for debts.

2 Model

For simplicity of exposition, we consider a three-date model, with a time line of 0, 1, and 2. One risk-free bond and one risky stock are available for trading. The bond is supplied elastically and the risk-free rate is taken to be zero. The stock is traded competitively in the stock market with the short-sales constraint, and its payoff at time 2 is V. We assume that the financial market is populated by two types of investors i and j with equal size of $1/2$, so the population size is normalized to one. At time 0, each investor is endowed with the per-capita supply of the stock, $x > 0$, and an initial cash position of w_0. Investors trade in the stock market at time 0 and time 1 and consume at time 2. They are risk averse and have a utility function of $u(w)$. We only require the utility function $u(w)$ to be twice continuously differentiable and

$$u'(w) > 0, \qquad u''(w) < 0.$$

Dynamic trading takes place because investors have heterogeneous beliefs about V, the terminal payoff of the stock at time 2. Figure I depicts the belief structure of investors. Two states exist at time 2: high state and low state. The stock payoff is V_H in the high state and V_L in the low state. Here, we require only $V_H > V_L$. At time 1, there is a signal with two possible realizations: A and B. For type $a \in \{i, j\}$ investors, A occurs with a probability of p^a. Conditional on A, V_H occurs with q_A^a, and conditional on B, V_H occurs with q_B^a. At time 0, type a investors believe that the stock payoff is V_H with a probability of μ^a, and V_L with a probability of $1 - \mu^a$. Hence, we have that

$$\mu^a = p^a q_A^a + (1 - p^a) q_B^a, \qquad a \in \{i, j\}.$$

If $\mu^i > \mu^j$, type i is unconditionally more optimistic than type j. Similarly, at time 1, in state A, type i is conditionally more optimistic than type j if $q_A^i > q_A^j$. The same statement holds for state B.

[FIGURE 1 about here.]

Let P_0, P_A, and P_B be the stock price at time 0, in state A, and state B at time 1, respectively. We define P_0^a, P_A^a, and P_B^a as the valuation of the stock by type a investors if all investors share the same beliefs as type $a \in \{i, j\}$. Following the spirit of Harrison and Kreps (1978), we define a negative bubble as follows.

Definition 1. *A negative bubble occurs at time* 0 *when the stock price under heterogeneous beliefs is lower than the lowest stock price that would obtain if all investors were homogeneous. That is,*

$$P_0 < \min_a \{P_0^a\}. \tag{1}$$

3 Numerical Examples

Before we present a formal proof for the existence of negative bubbles under the short-sales constraint, we first use numerical examples to illustrate how negative bubbles can arise when risk-averse investors trade dynamically. Specifically, we assume that investors have negative exponential utility function $u(w) = -e^{-\gamma w}$, where γ is the absolute risk aversion coefficient. We further assume that $V_H = 1$, $V_L = 0$, $x = 1$, and $\gamma = 1$.

3.1 The Static Model

To provide a benchmark, we first consider the static case in which trading is not allowed at time 1. At time 0, the probabilities that the stock pays off V_H at time 2 are $\mu_i = 0.5$ and $\mu_j = 0.475$ for investors of types i and j, respectively. Therefore, type i is more optimistic than type j.

To calculate the valuation of the stock for type i, P_0^i, we assume that all investors share the same belief as type i. At time 0, type i investors trade to maximize the expected utility over their wealth at time 2. Their maximization problem is then given by

$$\max_{D^i \geq 0} \left\{ \mu^i \times \left[u(w_0 + xP_0^i + D^i(V_H - P_0^i)) \right] + (1 - \mu^i) \times \left[u(w_0 + xP_0^i + D^i(V_L - P_0^i)) \right] \right\}, \quad (2)$$

where D^i, $w_0 + xP_0^i$, and $D^i(V - P_0^i), V \in \{V_H, V_L\}$ denote type i investors' demand for the stock, wealth, and capital gain from trading at time 0, respectively. Taking the derivative of the above expression with respect to D_0^i and arranging terms gives

$$P_0^i = \frac{\mu^i u'(w_0 + xP_0^i + D^i(V_H - P_0^i))V_H + (1 - \mu^i)u'(w_0 + xP_0^i + D^i(V_L - P_0^i))V_L}{\mu^i u'(w_0 + xP_0^i + D^i(V_H - P_0^i)) + (1 - \mu^i)u'(w_0 + xP_0^i + D^i(V_L - P_0^i))}. \quad (3)$$

Hence, the equilibrium price P_0^i is the expected value of type i investors weighted by their marginal utilities at time 2:

$$u'(w_0 + xP_0^i + D^i(V - P_0^i)), \quad V \in \{V_L, V_H\}.$$

This price equation (3) holds for a general utility function. Because all investors are homogeneous, at equilibrium, they hold an identical position of x, the per-capita supply, in the stock. Consequently, P_0^i can be written as

$$
\begin{aligned}
P_0^i &= \frac{\mu^i u'(w_0 + xP_0^i + x(V_H - P_0^i))V_H + (1-\mu^i)u'(w_0 + xP_0^i + x(V_L - P_0^i))V_L}{\mu^i u'(w_0 + xP_0^i + x(V_H - P_0^i)) + (1-\mu^i)u'(w_0 + xP_0^i + x(V_L - P_0^i))} \\
&= \frac{\mu^i u'(w_0 + xV_H)V_H + (1-\mu^i)u'(w_0 + xV_L)V_L}{\mu^i u'(w_0 + xV_H) + (1-\mu^i)u'(w_0 + xV_L)} \\
&= \underbrace{[\mu^i V_H + (1-\mu^i)V_L]}_{\text{Expectation of the payoff}} \underbrace{- \mu^i(1-\mu^i)(V_H - V_L)^2 \frac{(u'(w_0 + xV_L) - u'(w_0 + xV_H))/(V_H - V_L)}{\mu^i u'(w_0 + xV_H) + (1-\mu^i)u'(w_0 + xV_L)}}_{\text{Risk premium}}.
\end{aligned}
$$

Here, $\mu^i V_H + (1-\mu^i)V_L$ and $\mu^i(1-\mu^i)(V_H - V_L)^2$ are the expectation and variance of the final stock payoffs for type i investors at time 0. Hence, the price P_0^i can be expressed as the expectation of the payoff minus a risk premium.

Specially, when $u(w) = -e^{-\gamma w}$, we have

$$
P_0^i = \frac{\mu^i e^{-\gamma x V_H} V_H + (1-\mu^i)e^{-\gamma x V_L} V_L}{\mu^i e^{-\gamma x V_H} + (1-\mu^i)e^{-\gamma x V_L}} \tag{4}
$$

$$
= \underbrace{[\mu^i V_H + (1-\mu^i)V_L]}_{\text{Expectation of the payoff}} \underbrace{- \mu^i(1-\mu^i)(V_H - V_L)^2 \frac{(e^{-\gamma x V_L} - e^{-\gamma x V_H})/(V_H - V_L)}{\mu^i e^{-\gamma x V_H} + (1-\mu^i)e^{-\gamma x V_L}}}_{\text{Risk premium}}. \tag{5}
$$

The risk premium increases with risk aversion γ and is zero when $\gamma = 0$. When γ is small, the term

$$
\frac{(e^{-\gamma x V_L} - e^{-\gamma x V_L})/(V_H - V_L)}{\mu^i e^{-\gamma x V_H} + (1-\mu^i)e^{-\gamma x V_L}}
$$

can be approximated by γx.[5] We can then obtain the following approximate equation:

$$
P_0^i \approx [\mu^i V_H + (1-\mu^i)V_L] - \gamma x \mu^i(1-\mu^i)(V_H - V_L)^2, \tag{6}
$$

which holds for a small γ. If the stock's final payoff follows a normal distribution instead of a binomial distribution, then equation (6) holds exactly. The risk premium is proportional to the risk aversion, the supply of the stock, and the variance of the stock payoff.

[5]Because e^{-y} can be approximated by $1-y$ for a small y, $\frac{(e^{-\gamma x V_L} - e^{-\gamma x V_L})/(V_H - V_L)}{\mu^i e^{-\gamma x V_H} + (1-\mu^i)e^{-\gamma x V_L}}$ can be approximated by $\frac{\gamma x}{1 - \gamma x(\mu^i V_H + (1-\mu^i)V_L)}$, which is very close to γx when γ is small.

Substituting $\gamma = 1$, $x = 1$, $V_H = 1$, $V_L = 0$, and $\mu^i = 0.5$ into equation (4) gives

$$P_0^i = \frac{0.5 \times e^{-1\times1\times1} \times 1 + (1-0.5) \times e^{-1\times1\times0} \times 0}{0.5 \times e^{-1\times1\times1} + (1-0.5) \times e^{-1\times1\times1}} = 0.2689.$$

The approximate value of P_0^i given by equation (6) is $0.5 - 1 \times 1 \times 0.25 = 0.25$, which is close to the true value of 0.2689. Similarly, if all investors share the same belief as type j investors, then the stock price at time 0 is given by $P_0^j = 0.2497$.

Now, we proceed to calculate the equilibrium price P_0 in the case of heterogeneous beliefs. Notice that in the static model, the market is closed at time 1 and investors cannot trade at time 1. At time 0, type $a \in \{i, j\}$ investors trade to maximize their expected utility over wealth at time 2, their maximization problem is then given by

$$\max_{D^a \geq 0} \left\{ \mu^a \times \left[-e^{-\gamma(w_0 + xP_0 + D^a(V_H - P_0))} \right] + (1-\mu^a) \times \left[-e^{-\gamma(w_0 + xP_0 + D^a(V_L - P_0))} \right] \right\}. \tag{7}$$

The optimal demand is given by

$$D^a = \frac{1}{\gamma(V_H - V_L)} \left[\log \frac{\mu^a (V_H - P_0)}{(1-\mu^a)(P_0 - V_L)} \right]^+. \tag{8}$$

Here, $[y]^+$ is the positive part of y.

If the short-sales constraint does not bind for type i or j investors, the market clearing condition at time 0 is given by

$$x = \frac{1}{2} D^i + \frac{1}{2} D^j \tag{9}$$

$$= \frac{1}{2\gamma(V_H - V_L)} \left[\log \frac{\mu^i (V_H - P_0)}{(1-\mu^i)(P_0 - V_L)} \right] + \frac{1}{2\gamma(V_H - V_L)} \left[\log \frac{\mu^j (V_H - P_0)}{(1-\mu^j)(P_0 - V_L)} \right]. \tag{10}$$

Solving for P_0 from the market clearing equation (10), we obtain the equilibrium stock price at time 0:

$$P_0 = \frac{\sqrt{\mu^i \mu^j} e^{-\gamma x V_H} V_H + \sqrt{(1-\mu^i)(1-\mu^j)} e^{-\gamma x V_L} V_L}{\sqrt{\mu^i \mu^j} e^{-\gamma x V_H} + \sqrt{(1-\mu^i)(1-\mu^j)} e^{-\gamma x V_L}} \tag{11}$$

$$= \frac{\sqrt{0.5 \times 0.475} e^{-1\times1\times1} \times 1 + \sqrt{(1-0.5)(1-0.475)} e^{-1\times1\times0} \times 0}{\sqrt{0.5 \times 0.475} e^{-1\times1\times1} + \sqrt{(1-0.5)(1-0.475)} e^{-1\times1\times0}}$$

$$= 0.2592 \in [P_0^j, P_0^i] \equiv [0.2497, 0.2689].$$

In the equilibrium, the stock price is lower than both types of investors' expectations of the final stock payoff. Hence, the short-sales constraint does not bind for them. In this case, there exists a representative investor in the market whose beliefs determine the equilibrium price. The representative investor is endowed with w_0 in cash and x shares of the stock, and believes that the probability that the stock pays V_H at time 2 is given by

$$\sqrt{\mu^i\mu^j}/(\sqrt{\mu^i\mu^j}+\sqrt{(1-\mu^i)(1-\mu^j)}) \in [\mu_j, \mu_i].$$

The equilibrium prices are the same as if all investors share the same beliefs of the representative investor.

As another example, we assume that $\mu^i = 0.7$ and $\mu^j = 0.2$; the beliefs of investors in this case are more heterogeneous. Following the same procedure as above, the valuations of the stock for type i and j investors are obtained as 0.4619 and 0.0842, respectively. The short-sales constraint now binds for type j investors; that is, $D^j = 0$. From the market clearing condition (9), we have $D^i = 2x = 2$. That is, at equilibrium, the following holds:

$$D^i = \frac{1}{\gamma(V_H - V_L)}\left[\log\frac{\mu^i(V_H-P_0)}{(1-\mu^i)(P_0-V_L)}\right]^+ = 2,$$

$$D^j = \frac{1}{\gamma(V_H - V_L)}\left[\log\frac{\mu^j(V_H-P_0)}{(1-\mu^j)(P_0-V_L)}\right]^+ = 0.$$

Solving for P_0 from $D^i = 2$, we obtain the equilibrium stock price at time 0 as $P_0 = 0.2400$. In the absence of short-sales constraints, the equilibrium stock price at time 0 is 0.2193. Therefore, the stock price $P_0 = 0.2400$ *with* the short-sales constraint is larger than 0.2193, which is the price *without* the constraint. It is also larger than 0.0842, the smaller of valuations of the stock by type j investors. We can generalize the above results into the following two propositions. The general proof for the first proposition is given in the Appendix, and that for the second proposition is contained in the main text following that proposition.

Proposition 1. *In a static model with heterogeneous beliefs, the short-sales constraint always increases stock prices.*

This result formalizes Miller's (1977) argument that short-sales constraints keep more pessimistic investors out of the market and thus exert upward pressure on the stock price.[6] In a one-period model with heterogeneous beliefs such as Miller (1977) and our current model, no learning occurs, and hence the short-sales constraint does not change the beliefs of investors. The constraint prevents the pessimistic investors from trading in the market, and thus stock prices tend to reflect a more optimistic valuation than they otherwise would. Therefore, the constraint increases the expectation component in the stock price. It reduces the number of shares of the stock that the optimistic investors hold but does not change the volatility of the stock payoff, which decreases the risk-premium component. Consequently, the short-sales constraint always increases the stock price in Miller and our one-period model with heterogeneous beliefs.

We have shown that the short-sales constraint always increases the stock price by comparing the stock price in a model *with* the short-sales constraint and that in a model *without* the constraint. Similarly, by comparing stock prices under *homogeneous* beliefs and under *heterogeneous* beliefs, we are able to reach the conclusion that neither a bubble nor a negative bubble arises in a static model. The following proposition summarizes this result.

Proposition 2. *In a static model, neither bubbles nor negative bubbles exist .*

Proof: We stress that neither bubbles nor negative bubbles exist in the static model of Miller (1977), because the stock price is determined by the marginal utility weighted average valuation of the investors who participate in the stock market. On one hand, the weighted average valuation cannot be higher than that of the most optimistic investors. Otherwise, no investors would be willing to hold the stock, and consequently the market would not clear. As a result, the stock price has to decrease to below the valuation of the most optimistic investors. On the other hand, the weighted average valuation cannot be lower than that

[6] By contrast, in one-period rational expectations models with asymmetric information, Bai, Chang, and Wang (2006) and Cao, Zhang, and Zhou (2007) demonstrate that the short-sales constraint may decrease the equilibrium stock price. As equation (5) shows, the stock price has two components: the expectation of the final stock payoff and the risk premium. The risk premium is proportional to both the number of shares of the stock that marginal investors hold and the stock volatility. In these models, uninformed investors rationally learn from the stock prices to infer informed investors' private information, and the short-sales constraint prevents informed investors with negative views from trading in the market. Hence, it reduces the private information contained in the stock price, which increases the risk of the stock perceived by the less informed investors. As a result, the risk premium may increase as the volatility of the stock payoff increases, although the constraint reduces the number of shares of the stock that the marginal investors hold. The short-sales constraint does not change less informed investors' expectation of the stock payoff, because they are rational and learn from the stock price. Consequently, the short-sales constraint may decrease the stock price due to an increase in the risk premium.

of the most pessimistic investors either. Suppose that the short-sales constraint binds for the pessimistic investors; the stock price must then be higher than the pessimistic investors' expectation of the stock payoff. Otherwise, the pessimistic investors would buy some shares of the stock, or the short-sales constraint would not bind. Therefore, the stock price must be higher than the valuation of the most pessimistic investors.

3.2 The Dynamic Trading Model

We now consider a dynamic trading model in which investors are allowed to trade at time 1. We assume that investors have the following beliefs:

$$p^i = 0.25, \quad q_A^i = 0.2, \quad q_B^i = 0.6;$$

$$p^j = 0.65, \quad q_A^j = 0.65, \quad q_B^j = 0.15.$$

Note that the belief structure of investors is not symmetric in this example.

At time 0, the unconditional probabilities of V_H are given by

$$\mu^i = p^i q_A^i + (1 - p^i) q_B^i = 0.25 \times 0.2 + (1 - 0.25) \times 0.6 = 0.5$$

and

$$\mu^j = p^j q_A^j + (1 - p^j) q_B^j = 0.65 \times 0.65 + (1 - 0.65) \times 0.15 = 0.475$$

for investors of types i and j, respectively. Therefore, type i is unconditionally more optimistic than type j. $p^j > p^i$ means that at time 0, type j believes that state A occurs with a higher probability. At time 1, conditional on A, the probability that the stock pays V_H is higher for type j than for type i, and hence type j is more optimistic; conditional on B, the probability that the stock pays V_H is higher for type i than for type j, and hence type i is more optimistic. Consequently, the short-sales constraint may bind for type i investors in state A and bind for type j investors in state B.

If all investors share the same beliefs as type a, $a \in \{i, j\}$, no trading occurs at time 1, and in equilibrium, investors hold an identical position of x, the per-capita supply, in the stock. The equilibrium price P_0^a is the same as that in the static model. Hence, we have

$$P_0^i = 0.2689 \quad \text{and} \quad P_0^j = 0.2497.$$

We next demonstrate that the equilibrium stock price at time 0 can be lower than $\min\{P_0^i, P_0^j\} = 0.2497$, the lowest stock price that would obtain if all investors were homogeneous.

Applying a backward induction method and following the same procedure implemented in the static model, we first solve for the stock prices at time 1. In state A, type a investors' maximization problem is given by

$$\max_{D_A^a \geq 0} \left\{ q_A^a \times \left[-e^{-\gamma(w_A^a + D_A^a(V_H - P_A))} \right] + (1 - q_A^a) \times \left[-e^{-\gamma(w_A^a + D_A^a(V_L - P_A))} \right] \right\}, \tag{12}$$

where w_A^a and D_A^a denote type a's wealth and demand for the stock in state A, $a \in \{i, j\}$. The optimal demand is given by

$$D_A^a = \frac{1}{\gamma(V_H - V_L)} \left[\log \frac{q_A^a (V_H - P_A)}{(1 - q_A^a)(P_A - V_L)} \right]^+. \tag{13}$$

When state A is realized, type j investors are more optimistic and the short-sales constraint does not bind for them, but the constraint could bind for type i. When the short-sales constraint does bind for type i, from the market clearing condition (9), we have

$$D_A^i = \frac{1}{\gamma(V_H - V_L)} \left[\log \frac{q_A^i (V_H - P_A)}{(1 - q_A^i)(P_A - V_L)} \right]^+ = 0,$$
$$D_A^j = \frac{1}{\gamma(V_H - V_L)} \left[\log \frac{q_A^j (V_H - P_A)}{(1 - q_A^j)(P_A - V_L)} \right]^+ = 2.$$

Solving for P_A from $D_A^j = 2$, we obtain the equilibrium stock price in state A, $P_A = 0.2009$. It is straightforward to verify that the short-sales constraint indeed binds for type i. Similarly, in state B, we have $D_B^i = 2$, $D_B^j = 0$, and $P_B = 0.1687$.

At time 0, type i investors are more pessimistic because they believe that state A occurs with a lower probability, $p^i = 0.25 < 0.65 = p^j$, in which the stock is higher, $P^A = 0.2009 > 0.1687 = P^B$. Hence, to solve for the stock price at time 0, P_0, we first assume that the short-sales constraint binds for type i investors. Note that after purchasing the stock at time 0, type j investors will hold the stock until time 2 if state A realizes at time 1 and will sell it to type i investors at the price of P_B if state B realizes. Hence, for type j investors, three values of the stock payoff are possible: V_H, V_L, and P_B, with probabilities of $p^j q_A^j$, $p^j(1-q_A^j)$, and $1-p^j$, respectively. The optimization problem for type j investors is then given by

$$\max_{D_0^j>0} \Big\{ p^j q_A^j \left[-e^{-\gamma(w_0+xP_0+D_0^i(P_A-P_0)+2x(V_H-P_A))} \right] + p^j(1-q_A^j) \left[-e^{-\gamma(w_0+xP_0+D_0^i(P_A-P_0)+2x(V_L-P_A))} \right]$$

$$+(1-p^j) \left[-e^{-\gamma(w_0+xP_0+D_0^i(P_B-P_0))} \right] \Big\}. \tag{14}$$

Differentiating equation (14) with respect to D_0^j and arranging terms yields

$$\begin{aligned}
P_0 &= \frac{p^j q_A^j e^{-\gamma(D_0^i P_A+2x(V_H-P_A))} P_A + p^j(1-q_A^j) e^{-\gamma(D_0^i P_A+2x(V_L-P_A))} P_A + (1-p^j) e^{-\gamma D_0^i P_B} P_B}{p^j q_A^j e^{-\gamma(D_0^i P_A+2x(V_H-P_A))} + p^j(1-q_A^j) e^{-\gamma(D_0^i P_A+2x(V_L-P_A))} + (1-p^j) e^{-\gamma D_0^i P_B}} \\
&= \frac{p^j q_A^j e^{-2\gamma x V_H} P_A + p^j(1-q_A^j) e^{-2\gamma x V_L} P_A + (1-p^j) e^{-2\gamma x P_B} P_B}{p^j q_A^j e^{-2\gamma x V_H} + p^j(1-q_A^j) e^{-2\gamma x V_L} + (1-p^j) e^{-2\gamma x P_B}} \\
&\quad \text{(At equilibrium, } D_0^j = 2x.\text{)} \\
&= \frac{0.65 \times 0.65 \times e^{-2} \times 0.2009 + 0.65 \times 0.35 \times e^{0} \times 0.2009 + 0.35 \times e^{-0.3374} \times 0.1687}{0.65 \times 0.65 \times e^{-2} + 0.65 \times 0.35 \times e^{0} + 0.35 \times e^{-0.3374}} \\
&= 0.1859 < \min\{P_0^i, P_0^j\} = 0.2497.
\end{aligned}$$

That is, the stock price is lower than the lowest stock price that would obtain if all investors were homogeneous, leading to a negative bubble.[7]

With short-sales constraints and risk aversion, dynamic trading has two competing effects on the stock price: the expectation effect and the risk-premium effect. On one hand, the short-sales constraint prevents the pessimistic investors from participating in the stock market, and hence the constraint tends to increase the expectation of the final stock payoff. On the other hand, the optimistic investors have to hold the entire stock supply, so they demand a larger risk premium. Therefore, the existence of a negative bubble depends on whether the risk-premium effect dominates the expectation effect.

Why do negative bubbles exist in the dynamic trading model but not in the static one? The reason is as follows. In a *dynamic* trading model, investors have the option to sell the stock now and buy it back in the future, but they do not have the repurchase option in a

[7] Switching A with B and substituting j with i in equation (14), we obtain the optimization problem for type i investors. It can be verified that the optimal demand for them is negative given the stock price of $P_0 = 0.1859$. Hence, the short-sales constraint does bind for type i investors.

static model. It is then possible that a type of investors who are unconditionally optimistic are pessimistic in some future state that they believe will occur with a high probability. Therefore, some unconditionally optimistic investors might initially sell the stock, which drives down the stock price today. They hope to buy back the stock at an even lower price in the future, which would never happen in a static model.

In our dynamic model, for example, at time 1, in state B, type i investors are more optimistic and hold the entire supply of the stock; in state A, they are more pessimistic and hold a zero position in the stock. For type i investors, the benefit derived from selling a share of the stock at time 0 is $P_0 = 0.1859$, and the benefit from holding a share of the stock through time 1 will be $P_A = 0.2009 > P_0$ in state A and $P_B = 0.1687 < P_0$ in state B. At time 0, type i investors are more optimistic than type j; that is, $\mu^i = 0.5 > 0.475 = \mu^j$. However, they believe that state B at time 1 will occur with a relatively high probability and that the benefit from selling the stock exceeds that from holding the stock. Hence, type i investors choose to sell the stock at time 0, hoping to buy it back at a lower price at time 1. Consequently, at time 0, the pessimistic type j investors hold the entire stock supply and thus demand a higher risk premium. Recall that in a one-period model, only when the unconditional expectations of the two types of investors differ significantly, will the short-sales constraint be binding. In this case, the pessimistic type j investors sell the stock to the optimistic type i investors, who hold the entire stock supply.

In summary, in the example that we are investigating, both the expectation component and the risk-premium component in the equilibrium stock price in the our dynamic model are higher than those in the case of homogeneous beliefs in which all investors share the same beliefs as type j investors. But the increase in the risk premium surpasses the increase in expectation, leading to a negative bubble.

In Harrison and Kreps (1978), at time 0, investors buy the stock at a price above their valuation,[8] hoping to sell it at an even higher price at some future date. In our model, at time 0, the unconditionally more optimistic investors sell the stock at a price below their valuation, hoping to buy it back at an even lower price at time 1. Specifically, at time 0, type i investors sell the stock to type j investors at a price of $P_0 = 0.1859 < 0.2689 = P_0^i$, hoping

[8]Note that in Harrison and Kreps, investors are risk neutral and their valuation of a stock equals their expectation of the stock's future payoff.

to buy it back at a lower price $P_B = 0.1687$ in future state B. Note that type i investors believe that state B will occur with a higher probability.

Because the risk premium decreases with the risk aversion of investors, γ, the expectation effect may dominate the risk-premium effect for a smaller γ. That is, given the same belief structure, a negative bubble may not always exist when the risk aversion decreases. For example, in the above example, suppose that we reduce the risk aversion from 1 to 0.6 but keep other parameters unchanged. Applying equation (4), the valuation of the stock for type i investors is given by

$$P_0^i = \frac{0.475 \times e^{-0.6\times1\times1} \times 1 + (1-0.475) \times e^{-0.6\times1\times0} \times 0}{0.475 \times e^{-0.6\times1\times1} + (1-0.475) \times e^{-0.6\times1\times0}} = 0.3318.$$

Similarly, we have $P_0^j = 0.3543$.

We next calculate the stock price in the case of heterogeneous beliefs. In state A at time 1, the short-sales constraint binds for type i investors, so type j investors hold the entire stock supply. Hence, the equilibrium stock price is given by

$$P_A = \frac{0.65 \times e^{-0.6\times2\times1} \times 1 + (1-0.65) \times e^{-0.6\times2\times0} \times 0}{0.65 \times e^{-0.6\times2\times1} + (1-0.65) \times e^{-0.6\times2\times0}} = 0.3587.$$

Similarly, in state B, the short-sales constraint binds for type j and the stock price is given by $P_B = 0.3112$. At time 0, the constraint binds for type i and the stock price is given by

$$P_0 = 0.6325 - 0.2930 = 0.3395 > 0.3318 = 0.475 - 0.1432 = P_0^j.$$

In this example, $0.6325 - 0.475 = 0.1575 > 0.1498 = 0.2930 - 0.1432$; that is, the expectation effect dominates the risk-premium effect. Hence, a negative bubble does not exist for a smaller risk aversion of $\gamma = 0.6$.

As the risk aversion decreases further, a bubble may even arise. For example, when the risk aversion decreases to $\gamma = 0.2$, the stock prices at time 1 are $P_A = 0.5545$ and $P_B = 0.5014$, respectively; the stock price at time 0 is $P_0 = 0.5354$, which is larger than the higher of the two valuations of $P_0^i = 0.4502$ and $P_0^j = 0.4255$. When investors are risk neutral or $\gamma = 0$, a bubble always exists for an arbitrary heterogeneous belief structure, as shown by Harrison and Kreps (1978). The reason is that when $\gamma = 0$, the risk-premium effect vanishes, but the expectation effect remains. Hence, the equilibrium stock price is always higher than the highest valuation among investors, leading to a bubble.

Under the current belief structure, a bubble exists for a smaller risk aversion of $\gamma = 0.2$. However, even with a smaller risk aversion, we can find another heterogeneous belief structure under which a negative bubble arises. We next change the belief structure of investors to

$$p^i = 0.05, \quad q_A^i = 0.31, \quad q_B^i = 0.51;$$

$$p^j = 0.8, \quad q_A^j = 0.52, \quad q_B^j = 0.295.$$

Note that in the previous belief structure, we have $p^i = 0.25$, $p^j = 0.65$, $q_B^i = 0.6$, and $q_A^j = 0.65$. In the current belief structure, $q_B^i = 0.51$ and $q_A^j = 0.52$ are closer to the unconditional probability $\mu^j = 0.475$, $p^i = 0.05$ is closer to 0, and $p^j = 0.8$ is closer to 1. Now, the stock prices in states A and B at time 1 are given by $P_A = 0.4207$ and $P_B = 0.4110$, respectively. The stock price at time 0 is given by $P_0 = 0.4187$, which is smaller than the lower of the two valuations of $P_0^i = 0.4502$ and $P_0^j = 0.4255$, leading to a negative bubble.

For a risk aversion of $\gamma = 0.2$, a bubble exists under the previous belief structure:

$$p^i = 0.25, q_A^i = 0.2, q_B^i = 0.6; \quad p^j = 0.65, q_A^j = 0.65, q_B^j = 0.15.$$

But a negative bubble exists under the current belief structure:

$$p^i = 0.05, q_A^i = 0.31, q_B^i = 0.51; \quad p^j = 0.8, q_A^j = 0.52, q_B^j = 0.295.$$

Under both belief structures, at time 1, type j investors are more optimistic and hold the entire supply of the stock in state A; type i investors are more optimistic and hold the entire supply of the stock in state B. At time 0, type i investors who have a higher valuation ($P_0^i > P_0^j$) are more pessimistic ($p^i < p^j$) and hence sell the shares that they already own to type j investors.

Then why does a bubble exist under the previous belief structure but a negative bubble exists under the current one? The reason is as follows. Under the previous belief structure, at time 0, type j investors who have a lower valuation buy the stock at a price of $P_0 = 0.5354$, which is higher than their expectation of the stock payoff[9]:

$$\mu^j V_H + (1-\mu^j)V_L = 0.475 \times 1 + (1-0.475) \times 0 = 0.475.$$

They expect to sell the stock in some future state (state B in this example) at a price of $P_B = 0.5014$, which is higher than their expectation of the stock payoff:

$$q_B^j V_H + (1-q_B^j)V_L = 0.15 \times 1 + (1-0.15) \times 0 = 0.15.$$

That is, a resale option exists.

Under the current belief structure, at time 0, type i investors who have an unconditionally higher valuation sell the stock at a price of $P_0 = 0.4187$, which is lower than their valuation of the stock ($P_0^i = 0.4225$). They hope to buy back the stock in state B at an even lower price ($P_B = 0.4110 < P_0$). That is, a repurchase option exists. Note that type i investors believe that state B will occur with a high probability of $1 - p^i = 0.95$.

We have shown that given a belief structure, as investors' risk aversion decreases, the price can change from a negative bubble to a positive bubble. This finding is consistent with Harrison and Kreps (1978), who demonstrate that a positive bubble always exists when investors are risk neutral under short-sales constraints. We have also shown that even when investors' risk aversion decreases, we can find a belief structure such that a negative bubble occurs in the presence of short-sales constraints. Indeed, we show in the next section that for any risk-averse investors, there always exists a heterogeneous belief structure such that a negative bubble occurs. Although sidelined investors do not affect the expectation of investors who hold the stocks, they can still affect the stock price because the investors buying the stocks have to hold more stocks and thus demand a higher risk premium.

4 Negative Bubbles: The General Case

In this section, we show that with a general risk averse utility and dynamic trading, we can always find a heterogeneous belief structure that generates a negative bubble even under the short-sales constraint. For simplicity of the proof, we assume that the beliefs of investors of

[9]Note that risk-averse investors' valuation of a stock equals their expectation of the stock payoff minus a risk premium.

type i and j are symmetric; that is,

$$p^i = 1 - p, \quad q^i_A = 1 - q, \quad q^i_B = q,$$
$$p^j = p, \quad q^j_A = q, \quad q^j_B = 1 - q,$$
$$1/2 \leq p \leq 1, \quad 1/2 \leq q \leq 1.$$

Investors now have the same unconditional beliefs regarding the stock payoff; the unconditional probability of the stock paying V_H at time 2 is $\mu^i = \mu^j = pq + (1-p)(1-q)$ for both types of investors. At time 0, the probability with which type i investors believe that state A occurs is the same as the probability with which type j investors believe that state B occurs. At time 1, conditional on states A and B, the probability with which type i investors believe that the stock pays off V_H is the same as the probability with which type j investors believe that the stock pays off V_L. When $p = 1/2$ and $q = 1/2$, all investors share the same beliefs. When p and q grow, and the difference between the beliefs that type i and j investors possess for a certain state becomes larger. For example, at time 0, type i and j's beliefs of state A are $1 - p$ and p, the difference between their beliefs is $2p - 1$, which becomes larger as p grows. That is, investors' beliefs are more heterogeneous when p and q are larger. The symmetry in investors' conditional beliefs is not crucial, as demonstrated in the numerical examples presented in section 3, but it simplifies the analysis greatly.

We first solve for the stock price P_0^a that would obtain when all investors share the same belief of type a, $a \in \{i, j\}$. In our setting, if investors have the same beliefs, then even when the market reopens at time 1, no trading will occur, because investors are homogeneous (Milgrom and Stokey (1982)).

Investors of type a trade at time 0 to maximize their expected utility at time 2. Because investors are homogeneous, in equilibrium, each investor holds a position of x, the per-capita supply, in the stock. The equilibrium price P_0^a is type a investors' expectation of the stock's final payoff weighted by their marginal utility at time 2:

$$P_0^a = \frac{[pq + (1-p)(1-q)]u'(w_0 + xV_H)V_H + [p(1-q) + (1-p)q]u'(w_0 + xV_L)V_L}{[pq + (1-p)(1-q)]u'(w_0 + xV_H) + [p(1-q) + (1-p)q]u'(w_0 + xV_L)}. \quad (15)$$

Here, $pq + (1-p)(1-q)$ and $p(1-q) + (1-p)q$ are the unconditional probabilities of V_H and V_L; $w_0 + xP_0^a + x(V_H - P_0^a) = w_0 + xV_H$ and $w_0 + xV_L$ are investors' wealth at time 2 when the stock pays off V_H and V_L, respectively.

We now consider the case of heterogeneous beliefs. When state A realizes, type j is more optimistic and the short-sales constraint does not bind for them, but the constraint may bind for type i. When the short-sales constraint does not bind for type i, the first-order condition with respect to D_A^a, $a \in \{i, j\}$, is given by

$$q_A^a(V_H - P_A)u^{'}(w_A^a + D_A^a(V_H - P_A)) + (1 - q_A^a)(V_L - P_A)u^{'}(w_A^a + D_A^a(V_L - P_A)) = 0, \quad (16)$$

where $w_A^a = w_0 + xP_0 + D_0^a(P_A - P_0)$, which is type a investors' wealth in state A. The market clearing condition is given by

$$\frac{1}{2}D_A^i + \frac{1}{2}D_A^j = x, \qquad D_A^i > 0, D_A^j > 0. \quad (17)$$

To solve for the equilibrium price in state A, we first solve for demands $D_A^i(P_A)$ and $D_A^j(P_A)$ as functions of the stock price P_A from equation (16). Substituting $D_A^i(P_A)$ and $D_A^j(P_A)$ into the market clearing condition (17) gives the equilibrium price P_A, and the equilibrium demands come from the demand functions, $D_A^i(P_A)$ and $D_A^j(P_A)$.

Rewriting equation (16) gives

$$\begin{aligned} P_A &= \frac{q_A^a u^{'}(w_A^a + D_A^a(V_H - P_A))V_H + (1 - q_A^a)u^{'}(w_A^a + D_A^a(V_L - P_A))V_L}{q_A^a u^{'}(w_A^a + D_A^a(V_H - P_A)) + (1 - q_A^a)u^{'}(w_A^a + D_A^a(V_L - P_A))} \quad (18) \\ &< q_A^j V_H + (1 - q_A^j)V_L. \end{aligned}$$

Equation (18) means that the equilibrium price is type a investors' expectation of the stock's final payoff weighted by their marginal utility at time 2. The strict inequality holds because investors are strictly risk averse and type j investors are more optimistic in state A, they thus demand a premium for bearing risk from holding the stock. This finding implies that the equilibrium stock price is lower than type j investors' expectation of the stock final payoff.

When the short-sales constraint does bind for investor i, the stock price must be higher than his conditional expectation of the stock's final payoff; that is,

$$P_A > q_A^i V_H + (1 - q_A^i)V_L. \quad (19)$$

The first-order condition for type j investors does not change. The market clearing condition then becomes

$$D_A^i = 0, \qquad D_A^j = 2x. \tag{20}$$

Therefore, when pessimistic investors are sidelined, the remaining investors have to absorb the entire stock supply. They will then demand a higher risk premium, leading to a potentially lower stock price.

Depending on the specific values of p and q, the short-sales constraint may or may not bind in a state. To be comparable with the Harrison and Kreps (1978) model in which the short-sales constraint always binds for the pessimistic investors in each state, our analysis hereafter focuses on the case in which the short-sales constraint binds in states A and B. By the symmetry of the belief structure, the stock price in state B is the same as that in state A, which equals the stock price at time 0[10]:

$$P_0 = P_A = P_B.$$

Combining investor j's first-order condition with the market clearing condition (20) yields

$$P_0 = \frac{qu'(w_0 - xP_0 + 2xV_H)V_H + (1-q)u'(w_0 - xP_0 + 2xV_L)V_L}{qu'(w_0 - xP_0 + 2xV_H) + (1-q)u'(w_0 - xP_0 + 2xV_L)}. \tag{21}$$

From equation (21), we can solve for the equilibrium price P_0. As the right-hand side of equation (21) varies between V_L and V_H, a solution $P_0 \in [V_L, V_H]$ always exists.[11] In state A, investors of type j are more optimistic, and hence the short-sales constraint does not bind for them. Their wealth at time 2 is given by

$$w_0 + xP_0 + D_0^j(P_A - P_0) + D_A^j(V - P_A),$$

which simplifies to

$$w_0 - xP_A + 2xV, V \in \{V_L, V_H\},$$

because $D_0^j = 2x$ and $P_0 = P_A$ at equilibrium. Hence, the stock price P_A is again the expected value of type j investors weighted by their marginal utility at time 2, similar to equation (18) in the case in which the short-sales constraint is not binding.

[10] Note that the interest rate is assumed to be zero.

[11] When investors' utility function $u(w)$ is of increasing or constant absolute risk aversion, the right-hand side of equation (21) decreases monotonically with P_0, and the solution for P_0 is unique. If multiple solutions exist, we take the largest one as P_0.

We now compare the stock price in the case in which investors have heterogeneous beliefs with that in the case in which investors share the same belief as one of the two types of investors. In the former case, the stock price is determined by investors with more optimistic beliefs in each state, but they have to hold the entire stock supply. In the latter case, all investors participate in the market, but their unconditional probability of V_H is lower than that of the marginal investors in the case of heterogeneous beliefs. In the absence of risk aversion, dynamic trading always increases the stock price because in the intermediate date, pessimistic investors are sidelined and only optimistic investors participate due to short-sales constraints. With risk aversion, however, the nonparticipation of pessimistic investors can reduce the stock price due to an increase in the risk premium. We show that a set of p and q values always exists such that the risk-premium effect dominates and that the stock price in the case of heterogeneous beliefs is lower than the valuations among all investors, leading to a negative bubble. This result is summarized in the following theorem, whose proof is given in the Appendix.

Theorem 1. *Suppose that the short-sales constraint is binding and that investors are strictly risk averse. A $\bar{q}$ and a function of $\bar{p}(q)$ defined over $[\bar{q}, 1]$ always exists such that for any $q > \bar{q}$ and any $p > \bar{p}(q)$, the stock price is lower than the valuations of all investors.*

We can understand the intuition behind this theorem from numerical examples discussed in section 3. In a static model in which investors are prohibited from trading at time 1, the unconditionally optimistic investors, who at time 0 believe that the stock pays off V_H with a higher probability, buy the stock at time 0. The stock price tends to reflect the average valuation of the more optimistic investors; therefore, it is higher than the lowest valuations among investors, as shown in Proposition 1. With dynamic trading, in each state, the stock is held by investors who are optimistic in that state. At time 0, the pessimistic investors, who believe that at time 1 the state with a lower stock price will occur with a higher probability, choose to sell the stock at time 0, hoping to buy it back at an even lower price at time 1. Hence, at time 0, the risk-averse optimistic investors have to hold the entire supply of the stock. Consequently, they demand a higher risk premium, which may cause the stock price to be lower than the valuations of all investors, leading to a negative bubble. Note that Theorem 1 provides only sufficient conditions for the existence of negative bubbles, but it provides sufficient and necessary conditions for a negative bubble that arises in the presence of short-sales constraints.

The combination of the risk aversion and the option to sell the stock now and repurchase it later causes the stock price to drop. The reason is that if some investors believe that they can buy the stock at a lower price in the future when other investors will turn pessimistic and sell the stock, then they will sell the stock at time 0. Intuitively, suppose that $p = 1$; that is, type i investors believe state B and type j investors believe state A will occur for sure. Under heterogeneous beliefs, in state A, type j investors are more optimistic and are the marginal investors. They believe the probability of the stock paying off V_H at time 2 is q; similarly, this is true for type i investors in state B. Under homogeneous beliefs, at time 0, all investors are identical and also believe that the probability of the high state at time 2 is $q > 1/2$. Rearranging equation (15), the stock price at time 0 under homogeneous beliefs can be expressed as

$$P_0^i = P_0^j = [qV_H + (1-q)V_L] - q(1-q)(V_H - V_L)^2 \frac{(u'(w_0 + xV_L) - u'(w_0 + xV_H))/(V_H - V_L)}{qu'(w_0 + xV_H) + (1-q)u'(w_0 + xV_L)}.$$

Similarly, from equation (21), the stock prices in states A and B at time 1 under heterogeneous beliefs can be expressed as

$$\begin{aligned} P_A = P_B =& [qV_H + (1-q)V_L] \\ & - q(1-q)(V_H - V_L)^2 \frac{(u'(w_0 - xP_B + 2xV_L) - u'(w_0 - xP_B + 2xV_H))/(V_H - V_L)}{qu'(w_0 - xP_B + 2xV_H) + (1-q)u'(w_0 - xP_B + 2xV_L)}. \end{aligned} \tag{22}$$

Therefore, the expectation components are the same in the stock prices under both homogeneous and heterogeneous beliefs. However, under heterogeneous beliefs, only half of the investors participate in the stock market, whereas under homogeneous beliefs, all investors do. The marginal investors hold x shares of the stock in the case of homogeneous beliefs, but they hold $2x$ shares of the stock in the case of heterogeneous beliefs. Therefore, when investors are risk averse and hence their demand curve is downward sloping, the stock prices at time 1, $P_A = P_B$, are lower under heterogeneous beliefs than the stock price, $P_0^i = P_0^j$, under homogeneous beliefs.[12] Because we have that $P_0 = P_A = P_B$ and that the

[12]We need to prove that the risk-premium term is larger in $P_A = P_B$ than in $P_0^i = P_0^j$. Because $V_L < P_B < V_H$ and $u'' < 0$, we have that

$$u'(w_0 + xV_H) = u'(w_0 + xP_B + x(V_H - P_B)) > u'(w_0 + xP_B + 2x(V_H - P_B)) = u'(w_0 - xP_B + 2xV_H)$$

stock prices are continuous in p, when p is slightly smaller than 1, $P_0 < P_0^i = P_0^j$ still holds. That is, the stock price at time 0 will still be lower in the case of heterogeneous beliefs.

We emphasize that our result requires only risk aversion. When investors are risk averse, suppose that they believe that with a high probability, some participants will be pessimistic in certain future states. They would sell the stock now and hope to buy it back later when the more pessimistic investors leave the market, pushing the current price down. This result does not go through when investors are risk neutral. As in Harrison and Kreps (1978), with risk neutrality, the stock price in every state of nature is determined by the investor who has the highest valuation in that state. Pessimistic investors have no effect on the stock price, because the short-sales constraint binds for all but the most optimistic investors in the state, and these investors require no risk premium for holding the stock.

To obtain concrete values of p and q under which negative bubbles arise, we next consider a special case in which investors' utility function is of the negative exponential form. The investors' demand for the stock is then given by equation (13). To illustrate our point that negative bubbles can arise under the short-sales constraint, we here consider only the case in which the short-sales constraint binds at time 1. In state A, when $q = 1/2$, investors are equally optimistic and the equilibrium demands for both types are $D_A^i = D_A^j = x$. As q grows, type j investors are more optimistic than type i, and hence they demand more shares, $D_A^i < x$, $D_A^j > x$. As q grows further to a certain point $\bar{q}$, the equilibrium demand for type

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and

$$u'(w_0 + xV_L) = u'(w_0 + xP_B + x(V_L - P_B)) < u'(w_0 + xP_B + 2x(V_L - P_B)) = u'(w_0 - xP_B + 2xV_L),$$

which implies that $u'(w_0+xV_H)/u'(w_0+xV_L) > u'(w_0+xP_B+2x(V_H-P_B))/u'(w_0+xP_B+2x(V_L-P_B))$. Therefore, the following inequality holds:

$$\frac{(u'(w_0 + xV_L) - u'(w_0 + xV_H))}{qu'(w_0 + xV_H) + (1-q)u'(w_0 + xV_L)} = \left[\left(1 - \frac{u'(w_0 + xV_H)}{u'(w_0 + xV_L)}\right)^{-1} - q\right]^{-1}$$

$$< \left[\left(1 - \frac{u'(w_0 - xP_B + 2xV_H)}{u'(w_0 - xP_B + 2xV_L)}\right)^{-1} - q\right]^{-1} = \frac{(u'(w_0 - xP_B + 2xV_L) - u'(w_0 - xP_B + 2xV_H))}{qu'(w_0 - xP_B + 2xV_H) + (1-q)u'(w_0 - xP_B + 2xV_L)}.$$

This completes the proof.

i investors is $D_A^i = 0$ and for type j is $D_A^j = 2x$. That is,

$$D_A^i = \frac{1}{\gamma(V_H - V_L)} \left[\log \frac{\bar{q}(V_H - P_A)}{(1-\bar{q})(P_A - V_L)} \right] = 0,$$

$$D_A^j = \frac{1}{\gamma(V_H - V_L)} \left[\log \frac{(1-\bar{q})(V_H - P_A)}{\bar{q}(P_A - V_L)} \right] = 2x.$$

Solving the above equation yields

$$\bar{q} = \frac{1}{1 + e^{-\gamma x (V_H - V_L)}}. \tag{23}$$

Equation (23) shows that $\bar{q}$ increases with risk aversion γ. For a larger γ, to compensate the optimistic investors for absorbing the entire stock supply, their expectation of the stock payoff, $qV_H + (1-q)V_L$, has to be higher, which requires a higher q. Consequently, $\bar{q}$ increases with risk aversion γ.

The short-sales constraint binds at time 1 for a $q > \bar{q}$. For a given q, we need to find a $\bar{p}(q)$ such that a negative bubble arises when $p > \bar{q}(q)$. By definition, a negative bubble arises when the stock price at time 0 given in equation (21) is lower than the valuation of the stock given in equation (15). Combining this condition $P_0 < P_0^i = P_0^j$ with the assumption of $p \geq 1/2$, we have

$$\bar{p}(q) = \frac{1}{2} + \frac{1}{2q-1} \left[\frac{1}{2} - \left(1 + \frac{q}{1-q} e^{-\gamma x (V_H - V_L)} \right)^{-1} \right]^{+}. \tag{24}$$

Note that $\bar{p}$ decreases with risk aversion γ for a given q, and increases with q for a given γ.

The intuition behind equation (24) is best seen by using the approximate price equation (6). The difference between the equilibrium price in the case of heterogeneous beliefs, P_0, and that in the case of homogeneous beliefs, $P_0^i = P_0^j$, can be decomposed into two parts: the difference in the expectations and the difference in the risk premiums. The expectation component in P_0 is $qV_H + (1-q)V_L$[13] and that in P_0^i is $\mu^i V_H + (1-\mu^i)V_L$. The difference between these two expectations is given by

$$(q - \mu^i)(V_H - V_L) = (1-p)(2q-1)(V_H - V_L) \geq 0. \tag{25}$$

[13]This expression comes from equation (22) and the fact that $P_0 = P_A = P_B$.

Similarly, the difference between the two risk premiums is approximately given by

$$2\gamma x q(1-q)(V_H - V_L)^2 - \gamma x \mu^i (1-\mu^i)(V_H - V_L)^2 \tag{26}$$

$$= \left[\frac{1}{4} - \left(2 - 4\left(p - \frac{1}{2}\right)^2\right)\left(q - \frac{1}{2}\right)^2\right] \gamma x (V_H - V_L)^2 \geq 0, \tag{27}$$

which increases with p but decreases with q. Both differences are positive, or in other words, heterogeneous beliefs increase both the expectation and the risk premium in the stock price.

Hence, $P_0 < P_0^i$ holds or a negative bubble arises when the difference in the risk premiums dominates the difference in the expectations. For a smaller γ, the risk premium plays a less important role in determining the stock price, and the difference between the two risk premiums becomes smaller. For P_0 to be lower than P_0^i, the difference between the expectations (equation(25)) must be smaller, which requires a larger p. This means that $\bar{p}$ decreases with the risk aversion γ.

[FIGURE 2 about here.]

Figure II plots the stock price at time 0 as a function of γ in the cases of both homogeneous and heterogeneous beliefs. From equation (23), we know that the short-sales constraint binds at time 1 for

$$\gamma < \bar{\gamma} = \frac{\ln(q/(1-q))}{x(V_H - V_L)} = \frac{\ln(0.56/(1-0.56))}{1 \times (1-0)} = 0.2412,$$

and the constraint does not bind for $\gamma > \bar{\gamma}$. Hence, the stock price under the heterogeneous beliefs has a "kink" at $\bar{\gamma}$. From equation (24), we have that $\bar{p} = 1/2$ when $\gamma = \bar{\gamma}$. Therefore, a negative bubble by Theorem 1 arises. Because $\bar{p}$ decreases with γ for a given q, a negative bubble still arises when γ decreases from $\bar{\gamma}$ to a point such that $\bar{p} = p = 0.56$. Note that negative bubbles also arise for $\gamma > \bar{\gamma}$ when the short-sales constraint does not bind. This confirms that Theorem 1 provides sufficient conditions for the existence of negative bubbles.

The risk-premium component in the equilibrium stock price is proportional to the variance of the stock's final payoff. When $q \geq 1/2$ increases, the variance of the stock's final payoff decreases,[14] which in turn reduces the risk-premium component in the stock price. Conse-

[14]Note that the variance of the stock's final payoff is $q(1-q)(V_H - V_L)^2$ in P_0 and is $\mu^i(1-\mu^i)(V_H - V_L)^2$ in P_0^i. Both of them decrease with q when $q \geq 1/2$.

quently, the difference between the two risk premiums (equation(27)) decreases. On the other hand, p needs to be larger to keep the difference between the expectations (equation(25)) sufficiently small so that $P_0 < P_0^i$ holds. This means that $\bar{p}(q)$ increases with q.

Substituting $x = 1$, $V_H = 1$, $V_L = 0$, and $\gamma = 1$ into equations (23) and (24), we have

$$\bar{q} = \frac{1}{1 + e^{-1\times 1\times(1-0)}} = 0.7311, \quad \bar{p}(q) = \frac{1}{2} + \frac{1}{2q-1}\left[\frac{1}{2} - \left(1 + \frac{q}{1-q}e^{-1}\right)^{-1}\right]^{+}.$$

It is straightforward to calculate that

$$\bar{p}(0.75) = 0.5493 \quad \text{and} \quad \bar{p}(0.9) = 0.8350.$$

Similarly, if the risk aversion reduces to $\gamma = 0.2$, we have

$$\bar{q} = 0.5498, \quad \bar{p}(0.6) = 0.7559, \quad \text{and} \quad \bar{p}(0.75) = 0.9213.$$

[FIGURE 3 about here.]

Note that a negative bubble arises for $q > \bar{q}$ and $p > \bar{p}(q)$.[15] The above examples show that a negative bubble can arise under relatively mild conditions. For example, we have that $\bar{q} = 0.7311$, $\bar{p}(0.75) = 0.5493$ for $\gamma = 1$ and $\bar{q} = 0.5498$, $\bar{p}(0.6) = 0.7559$ for $\gamma = 0.2$, respectively. That is, a negative bubble arises for $p = 0.76$ and $q = 0.6$ when $\gamma = 0.2$. As the risk aversion γ decreases, $\bar{q}$ decreases but $\bar{p}(q)$ increases for a fixed q. This means that it is more difficult for a negative bubble to arise with a smaller risk aversion. As γ goes to 0, $\bar{p}(q)$ goes to 1 for any given $q > \bar{q}$. In other words, when investors are risk neutral or $\gamma = 0$, we have $\bar{p}(q) \equiv 1$. Therefore, no p exists such that $\bar{p}(q) < p \le 1$ holds, which means, by Theorem 1, that it is not possible to find a belief structure such that a negative bubble arises in the presence of short-sales constraints. In fact, when $\gamma = 0$, a bubble always arises for any heterogeneous belief structure, as shown by Harrison and Kreps (1978).

In Figure III, we plot the stock price as a function of p in the cases of both homogeneous and heterogeneous belief cases. Because $q = 0.56 > \bar{q} = 0.5498$, the short-sales constraint binds at time 1, and hence the stock price at time 0 is determined by equation (21) and does not vary with p. Compared with the stock price under homogeneous beliefs, by Theorem 1, the stock price under heterogeneous beliefs is higher for $p < \bar{p} = 0.5857$ and lower for $p > \bar{p}$.

[15] Recall that p and q measure the heterogeneity of the investors' beliefs. The larger p and q, the more heterogeneous the investors' beliefs.

5 Conclusion

Given the seminal work of Harrison and Kreps (1978), a conventional wisdom seems to be that a combination of short-sales constraints and heterogeneous beliefs always leads to a bubble. This paper demonstrates that the assumption of risk neutrality is crucial for the existence of bubbles. We show that, with risk aversion and the short-sales constraint, dynamic trading has two effects on the stock price. The short-sales constraint prevents the conditionally pessimistic investors from participating in the stock market, and consequently the conditionally optimistic investors have to hold the entire stock supply. As a result, the exclusion of the pessimistic investors enhances the expectation component in the stock price, but the risk-averse optimistic investors demand a larger risk premium. The first effect increases the stock price, but the second effect due to less risk sharing lowers the price. When investors have divergent views about the final stock payoffs in the intermediate states, the second effect can dominate, leading to a negative bubble. Under the assumption of risk neutrality, the second effect is absent, and hence the stock price is always higher in the presence of heterogeneous beliefs.

Whereas Harrison and Kreps develop a notion of resale options, this paper develops a notion of repurchase options. When investors believe that others will become optimistic in the future, they will buy the stock now, driving up the stock price today and hoping to sell it at a higher price later. When investors believe that others will become pessimistic in the future, they will sell the stock now, driving down the stock price today and hoping to buy it back at a lower price later. The relative importance of the resale option and the repurchase option depends on the investors' beliefs about the beliefs of other investors. Our model captures Keynes' (1936) notion that mass psychology in the future will have an impact today.

The resale option and the repurchase option are necessary for the existence of bubbles and negative bubbles. In a static trading model in which these options are absent, we illustrate that neither bubbles nor negative bubbles arise.

Appendix

Proof of Theorem 1

Under the symmetric belief structure, the valuations of type i and j investors are the same. Denote the stock prices at time 0 in the cases of homogeneous and heterogeneous beliefs as P_0^{Ho} and P_0^{He}, and define function

$$F(P;p,q,z) = 1 + \frac{p(1-q)+(1-p)q}{pq+(1-p)(1-q)} \frac{u'(w_0+xP+z(V_L-P))}{u'(w_0+xP+z(V_H-P))} - \frac{V_H-V_L}{P-V_L}. \tag{28}$$

We will show that with the short-sales constraint, a set of p and q exists such that $P_0^{He} < P_0^{Ho}$.

Because the utility function u is twice continuously differentiable, function F is continuously differentiable with respect to its argument P, p, q, and z. It can be verified that

$$F(V_H;p,q,z) > 0, \quad \lim_{P \to V_L} F(P;p,q,z) = -\infty. \tag{29}$$

Hence, a $P \in (V_L, V_H)$ exists such that $F = 0$. By the implicit function theorem, $F = 0$ defines a function $P = P(p,q,z)$, which is continuously differentiable in p, q, and z.

The stock price at time 0 in the case of homogeneous beliefs, P_0^{Ho}, and the stock price at time 0 in the case of heterogeneous beliefs, P_0^{He}, are determined by

$$F(P_0^{Ho};p,q,x) = 0, \quad F(P_0^{He};1,q,2x) = 0, \tag{30}$$

and can be expressed as

$$P_0^{Ho} = P(p,q,x), \quad P_0^{He} = P(1,q,2x). \tag{31}$$

Differentiating F with respect to P, p, q, and z gives

$$\frac{\partial F}{\partial P} = \frac{p(1-q)+(1-p)q}{pq+(1-p)(1-q)} \left(\frac{-u''(w_L)}{u'(w_H)} - \frac{-u''(w_H)u'(w_L)}{[u'(w_H)]^2} \right)(z-x) + \frac{V_H - V_L}{(P-V_L)^2}, \tag{32}$$

$$\frac{\partial F}{\partial p} = \left(\frac{1-2q}{pq+(1-p)(1-q)} - \frac{(p(1-q)+(1-p)q)(2q-1)}{(pq+(1-p)(1-q))^2} \right) \frac{u'(w_L)}{u'(w_H)}, \tag{33}$$

$$\frac{\partial F}{\partial q} = \left(\frac{1-2p}{pq+(1-p)(1-q)} - \frac{(p(1-q)+(1-p)q)(2p-1)}{(pq+(1-p)(1-q))^2} \right) \frac{u'(w_L)}{u'(w_H)}, \tag{34}$$

$$\frac{\partial F}{\partial z} = \frac{p(1-q)+(1-p)q}{pq+(1-p)(1-q)} \left(\frac{-u''(w_L)(P-V_L)}{u'(w_H)} + \frac{-u''(w_H)u'(w_L)(V_H-P)}{[u'(w_H)]^2} \right). \tag{35}$$

Here, $w_L \equiv w_0 + xP + z(V_L - P)$ and $w_H \equiv w_0 + xP + z(V_H - P)$.

We first consider an extreme case of $p = 1$, in which type i investors believe that only state A is possible, and type j investors believe that only state B is possible. We know that $\partial F(P; 1, q, x)/\partial P$ is continuous in q and $\partial F(P; 1, 1, x)/\partial P = \frac{V_H - V_L}{(P-V_L)^2} > 0$, so a $\bar{q}_1$ exists such that $\partial F(P; 1, q, z)/\partial P > 0$ for any $P \in (V_L, V_H), z \in [x, 2x]$ when $q \in (\bar{q}_1, 1)$.[16] From equation (35), we know $\frac{\partial F}{\partial z} > 0$. As a result, the implicit function theorem yields

$$\frac{\partial P(1, q, z)}{\partial z} = -\frac{\partial F}{\partial z} \Big/ \frac{\partial F}{\partial P} < 0, \quad q \in (\bar{q}_1, 1), \tag{36}$$

and

$$P^{He} - P^{Ho} = P(1, q, 2x) - P(1, q, x) = \int_x^{2x} \frac{\partial P(1, q, z)}{\partial z} dz < 0.$$

Thus, the stock price at time 0 in the case of heterogeneous beliefs is lower than the stock price at time 0 in the case of homogeneous beliefs, resulting in a negative bubble.

We now consider the general case $p \in [1/2, 1]$. We start with

$$\begin{aligned} P_0^{He} - P_0^{Ho} &= P(1, q, 2x) - P(p, q, x) \\ &= [P(1, q, x) - P(p, q, x)] + [P(1, q, 2x) - P(1, q, x)]. \end{aligned}$$

Heterogeneous beliefs exhibit two effects on the stock price. The first effect is that the short-sales constraint prevents the pessimistic investors from participating in the stock market and

[16] The uniqueness of P is not generally guaranteed. However, in the case of $p = 1$, $\partial F/\partial P > 0$ always holds, which ensures that the stock price is unique.

tends to increase the price, implying that $P(1,q,x) - P(p,q,x) > 0$.[17] The second effect is that the risk-averse optimistic investors have to hold the entire stock supply, and thus they demand a larger risk premium, which implies that $P(1,q,2x) - P(1,q,x) < 0$. P is continuous in p, so a $p \in (\bar{p}(q), 1]$ exists such that the second effect dominates the first one, and hence $P_0^{He} < P_0^{Ho}$. We here use $\bar{p}(q)$ to denote that this critical value may depend on q.

We now consider the restriction on parameters such that the short-sales constraint binds for type j investors in state A. In inequality (19), the left-hand side is increasing in q and the right-hand side is decreasing in q.[18] And $P_A = V_H$ when $q = 1$, so some $\bar{q}_2 \in (0,1)$ exists such that inequality (19) is satisfied when $q \in (\bar{q}_2, 1)$. Let $\bar{q} = \max\{\bar{q}_1, \bar{q}_2\}$; then a negative bubble arises if $q \in (\bar{q}, 1)$, $p \in (\bar{p}(q), 1]$, which also implies that the more heterogeneous the belief is, the more likely negative bubbles are to arise. Q.E.D.

Proof of Proposition 1

Consider two economies, C and NC, which are identical except that the short-sales constraint exists in economy C and the constraint does not exist in economy NC. We will show that the stock price P_0^C in economy C is higher than the stock price P_0^{NC} in economy NC.

Without loss of generality, we assume that $\mu^i > \mu^j$; that is, type i investors are more optimistic than type j investors. If the short-sales constraint does not bind in economy C, then we have $P_0^C = P_0^{NC}$. When the short-sales constraint binds for the pessimistic type j investors in economy C, from the market clearing condition (9), we have $D^j = 0$ and $D^i = 2x$ at equilibrium. Note that, in economy NC, we have $D^j < 0$ and $D^i > 2x$.

Consider type i investors, who trade to maximize the expected utility over their wealth at time 2, and their optimization problem is given by equation (8). From the first-order condition for type i's demand D^i, the stock prices in economies C and NC can be expressed

[17] $\frac{\partial P(p,q,z)}{\partial p} = -\frac{\partial F}{\partial p} \Big/ \frac{\partial F}{\partial P} > 0$, when $q \in (\bar{q}_1, 1)$ as $\frac{\partial F}{\partial p} < 0$ and $\frac{\partial F}{\partial P} > 0$.

[18] Note that P_0^{He} increases with q by the symmetry of p and q and $P_A = P_0^{He}$.

as

$$P_0^C = \frac{\mu^i u'(w_0 + xP_0^C + 2x(V_H - P_0^C))V_H + (1-\mu^i)u'(w_0 + xP_0^C + 2x(V_L - P_0^C))V_L}{\mu^i u'(w_0 + xP_0^C + 2x(V_H - P_0^C)) + (1-\mu^i)u'(w_0 + xP_0^C + 2x(V_L - P_0^C))},$$

$$P_0^{NC} = \frac{\mu^i u'(w_0 + xP_0^{NC} + D^i(V_H - P_0^{NC}))V_H + (1-\mu^i)u'(w_0 + xP_0^{NC} + D^i(V_L - P_0^{NC}))V_L}{\mu^i u'(w_0 + xP_0^{NC} + D^i(V_H - P_0^{NC})) + (1-\mu^i)u'(w_0 + xP_0^{NC} + D^i(V_L - P_0^{NC}))}.$$

Following a similar procedure in the proof of Theorem 1, we define an implicit function:

$$F(P, y) = 1 + \frac{\mu_i u'(w_0 + xP + y(V_H - P))}{(1-\mu_i)u'(w_0 + xP + y(V_L - P))} - \frac{V_H - V_L}{V_H - P}.$$

By the implicit function theorem, $F = 0$ defines a function $P = P(y)$, which is continuously differentiable in y. The stock price at time 0 in economy C, P_0^C, and the stock price in economy NC, P_0^{NC}, are determined by

$$F(P_0^C; 2x) = 0, \quad F(P_0^{NC}; D^i > 2x) = 0, \tag{37}$$

and can be expressed as

$$P_0^C = P(2x), \quad P_0^{NC} = P(D^i > 2x). \tag{38}$$

It can be verified that $\partial P/\partial y < 0$. Hence, we have $P_0^C > P_0^{NC}$. Q.E.D.

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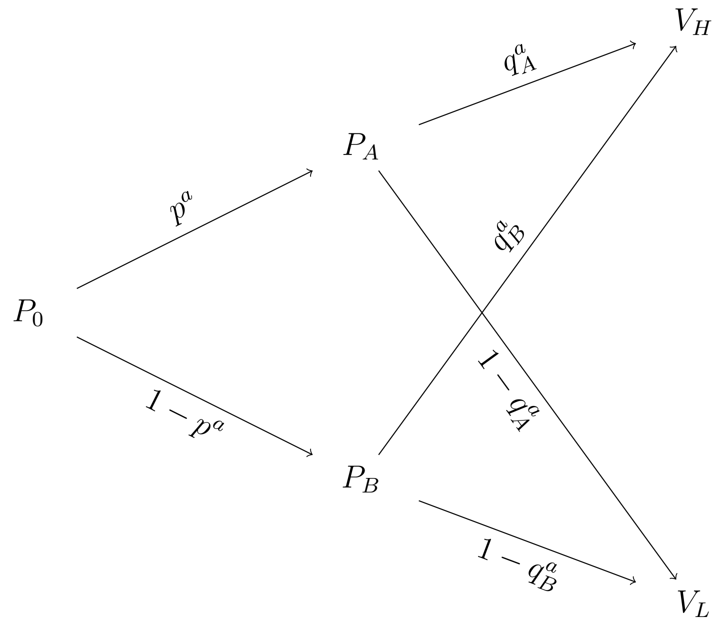

FIGURE I
Belief Structure

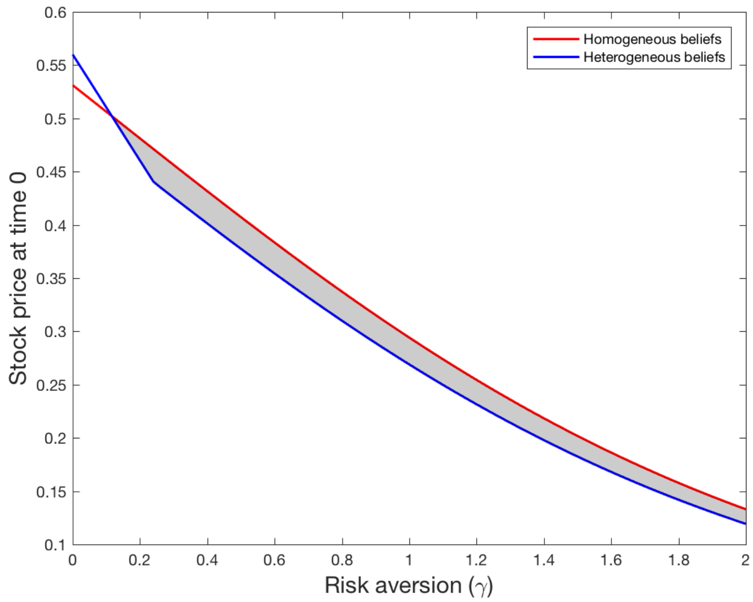

FIGURE II
Stock price as a function of risk aversion (γ)

Stock price as a function of risk aversion (γ) under homogeneous and heterogeneous beliefs. $V_H = 1$, $V_L = 0$, $x = 1$, $p = 0.56$, and $q = 0.76$.

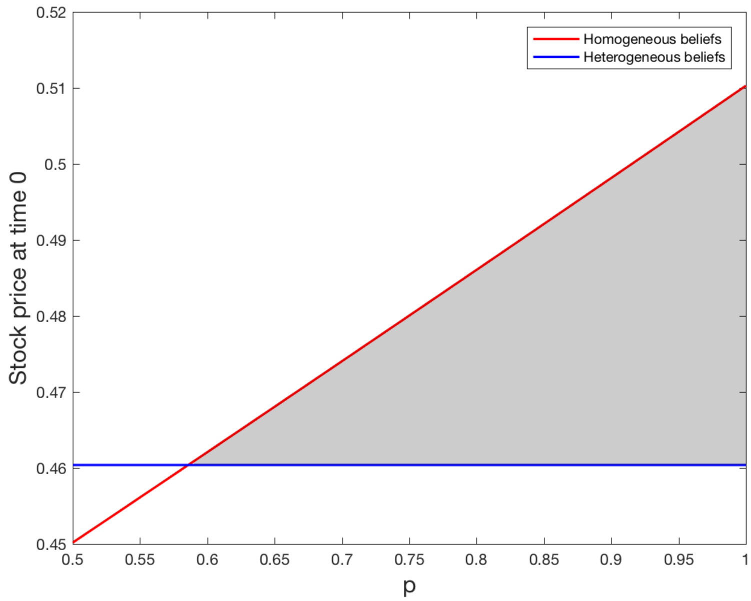

FIGURE III
Stock price as a function of p

Stock price as a function of p under homogeneous and heterogeneous beliefs. $V_H = 1$, $V_L = 0$, $x = 1$, $\gamma = 0.2$, and $q = 0.56$.